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THE EFFECT OF RISK AND UNCERTAINTY IN ECONOMIC ANALYSES OF INVESTMENTS IN CAPITAL ASSETS

A THESIS

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SUMMARY

The objectives of this study are threefold. The first objective is to explore the effects of random variation of important variables (elements) on the calculated results of capital investment economic analyses. Secondly, an objective is to develop decision guides for recognizing situations in which the consideration of random variation of those elements is advantageous. The final objective is to develop procedures and recommendations for the effective consideration of random variation in those economic analyses.

The study is concentrated on the consideration of risk and uncertainty when mutually exclusive projects are being compared and when non-mutually exclusive projects are being screened for acceptance or rejection. The term "project" is used to mean an investment alternative which is separable for economic analysis purposes. In the quantitative studies, the elements which are taken into account are the investment, project life, salvage value, and annual uniform net operating receipts or disbursements. Further, investments at a single point in time only are considered so that the effect of present decisions on future investment alternatives or vice-versa are not directly taken into account.

A comprehensive survey of various existing or recommended procedures and approaches for the consideration of risk and uncertainty in capital investment economic analyses is presented. Comments are made on advantages and limitations of each of these approaches.

An extensive analysis of the effect of various statistical distri-

butions of project life on expected values of key factors in economic analyses is made. The key factors so considered are the capital recovery factor and the uniform series present worth factor. Relative comparisons for each factor are presented by graphs showing ratios of the mathematical expectations based on consideration of the life distribution to the values of the factors at point estimates of the lives. The point estimated lives considered are usually expected lives, but modal lives are also used for some conditions. Life distributions considered include the normal, uniform, gamma, beta, and triangular distributions. A range of variation is considered for each distribution, with the coefficient of variation usually less than 40 per cent. Comparisons are made at interest rates of 5, 10, and 20 per cent. Most resulting ratios are graphed over a continuum of expected or modal lives between 2 years and 30 years.

A rather extensive graphical presentation is made of the effect of various salvage value functions on the capital recovery cost expressed as a proportion of the initial investment. Initial, or lump, drops in salvage values together with both straight line and exponential declines over project life for various interest rates and minimum salvage values are considered.

Graphical explorations of the sensitivity of the present worth of a project to changes in the amounts of the uniform periodic receipts or disbursements, life of project, interest rate, and salvage value are made for a wide range of project lives and interest rates. Also, formulas are presented for approximating the mean and variance of a measure of merit (present worth) of a project when variation of multiple elements is taken into account. Using these formulas, step-by-step procedures for comparing mutually exclusive projects and also non-mutually exclusive projects are developed.

The potential effects on analysis results caused by lack of certainty concerning the distribution of individual elements is examined. The extent of the effect of lack of certainty concerning the distribution of individual elements in such an analysis depends upon the circumstances inherent in the project or projects being analyzed. The major circumstances affecting the importance of that lack of certainty are: the relative importance of each element to the overall analysis outcome, the closeness of the decision if based on initial estimates of the respective element distributions, the degree of lack of certainty concerning those distributions, and the effect of variation of combinations of elements.

Sequential qualitative decision guides are presented for determining the extent to which an economic analysis should be pursued. These guides include consideration of when variation of elements should be taken into account by risk analyses involving estimated distributions for individual elements and also when lack of certainty concerning those estimated distributions should be examined. Extensive figures and graphs are included to aid the analyst in determining certain quantities of interest when considering variation of elements in the economic comparison of projects.

A digital computer program developed to aid in performing sensitivity studies for individual projects is presented. The program can be used to determine analysis outcomes when each element in any combination of one or more elements varies a certain number of standard deviations in a favorable or unfavorable direction from the expected values of those elements.

Also developed are particular applications of Robert Schlaifer's[1] expected opportunity loss information to value the decreased lack of certainty which can potentially be attained through added estimation study in economic analyses.

A breakeven method for considering lack of certainty concerning the distribution of project life is shown, and special adaptations of decisions rules for complete uncertainty are presented. Also, formulas which take into account the effects of discrete external conditions in the estimation of the mean and variance of a variable are developed.

Some statistical aspects of estimation for economic analyses are examined. Uses of Bayesian statistics in adjusting probability estimates are shown, and procedures for performing subjective estimations are recommended.

Finally, recommendations are made regarding further study to extend the results of this research and to test for practicability the procedures and graphical aids developed.

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1. Robert Schlaifer, *Probability and Statistics for Business Decisions,* McGraw-Hill, New York, 1959.

GLOSSARY OF SYMBOLS

The symbols which are used consistently in several sections or in figures are listed below.

A	Coefficient of risk aversion.
AW($)	Annual worth or cost.
C.AW($)	AW($) for assumed certainty (at expected values for all elements).
C.CRC($)	CRC($) as assumed certainty (at expected values for all elements).
Cov(T,S)	Covariance between elements T and S.
CRC	Capital recovery cost factor = CRC($)/P.
CRC_t	Capital recovery cost factor which depends on S_t.
CRC($)	Capital recovery cost.
CRF	Capital recovery factor = $i/1-e^{-iT}$.
D	Annual uniform net cash receipts or disbursements.
D_t	Annual net cash receipts or disbursements for year t.
E( )	Expected value of ( ).
f(T)	Density function of life, T
i	Interest rate (nominal annual).
LMS	Life at the point where the salvage value becomes a minimum.
P	First cost, or initial investment.
PDROP	Lump drop in salvage value which occurs at time investment is made.
PV($)	Present value.
$PV(\$)_d$	Difference between two projects (expressed in PV($)).

P( )	Probability of the occurrence of ( ).
R	E(PV)/PV at expected life.
R_m	E(PV)/PV at modal life.
R_{cr}	E(CRF)/CRF at expected life.
R_{crm}	E(CRF)/CRF at modal life.
ρ	Coefficient of correlation.
ρ_{st}	Coefficient of correlation of cash flows for years s and t.
S	Salvage value (constant).
S_t	Salvage value (varies as a function of T).
S_m	Minimum salvage value.
σ	Standard deviation.
σ_t^2	Variance of cash flows for year t (could be denoted $V(D_t)$).
σ_{st}	Covariance of cash flows for years s and t.
t	Year (general term).
T	Life of project or series (in years).
V( )	Variance of ( ).
$V_D(PV(\$))$	Contribution to V(PV($)) caused by variance of D.
W_ρ	$V_D(PV(\$))/\sigma_s^2$ where σ_s^2 is the variance of D_s for the s^{th} year and ρ is the coefficient of correlation for all ρ_s's.

CHAPTER I

INTRODUCTION

Objectives of Study

The objectives of this study are threefold. The first objective is to explore the effects of random variation of important variables (elements) on the calculated results of capital investment economic analyses. Secondly, an objective is to develop decision guides for recognizing situations in which the consideration of random variation of those elements is advantageous. The final objective is to develop procedures and recommendations for the effective consideration of random variation in those economic analyses.

Importance of Topic

The subject of capital planning, budgeting, and management is of critical importance to the health of a business enterprise. In total, this subject represents the basic top management function of the enterprise; the investment of funds in those activities where they will be most productive in promoting the profitability, long-range growth, and any other objectives of the enterprise.

A most important and probably the greatest effort-consuming aspect of capital planning and budgeting is the making of economic analyses of alternative capital investment projects to obtain quantitative measures of prospective investment worth which can aid in making the best choices between alternatives.

Unfortunately, these economic analyses are often beset with the inherent difficulties of estimating the elements which are vital in measuring worth. Examples of such elements are: initial investment required, life of investment, prospective salvage value, and periodic operating results (net receipts or disbursements). Another way of stating the problem is that some or all of these elements are random variables, thereby causing difficulty in the making of economic analyses which realistically take into account that variation. Harry Roberts (75-p.16) stated the problem and the importance of the problem very well in 1957:

> The most serious deficiency in the present state of knowledge about capital budgeting is the absence of a satisfactory framework for incorporating uncertainty into the analysis. Much of the ultimate success or failure of analytical methods in aiding or superseding intuitive methods of capital budgeting will hinge on future developments in the treatment of uncertainty.

Distinction Between Risk and Uncertainty

The usual distinction between risk and uncertainty as used in the current literature[1] is that a situation is characterized by risk only if the probabilities of the alternative, possible outcomes are known, while a situation is characterized by uncertainty if the probability distribution of the possible outcomes is not known.

Some writers, such as Farrar (27-pp.1-2), in subscribing to the above differentiation extend the distinction slightly by saying that to qualify as a risk situation an experiment must be repetitive in nature and must possess a probability density from which inferences can be made

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1. See Morris (65-pp.11-12), Heady (39), and Farrar (27-pp.1-2).

by objective statistical procedures. In this research, the distinction will not be so stringent, for subjective probabilities will be used to describe risk situations.

Barish (4-p.307) makes a less restrictive, broader differentiation:

> For most purposes, there is no need to distinguish between the meanings of the words "risk" and "uncertainty" and they can be used interchangeably. When we want to consider how to deal explicitly with risk and uncertainty, however, there is an advantage to more careful definition. We shall define risk as the dispersion of the probability distribution of the event whose value is being predicted. Uncertainty will be measured by the degree of lack of confidence that the estimated probability distribution is correct.

It is apparent that it is not possible to adopt a differentiation which will harmonize with all the current literature. However, Barish's broad differentiation seems to best meet the needs of research involving the effect of the variation of elements that are important to an economic analysis of capital investments. Usually in this research, when effects of variation of given elements are considered, specific probability distribution shapes or parameters will be used even though there is uncertainty as to the correct or most appropriate distribution shapes or parameters. Hence, this research will be primarily concerned with assumed risk problems even though in general the distributions used are not strictly determinable.

Interest Rates for Economic Analyses

All the research herein assumes known, constant interest rates in the calculation of measures of merit. These interest rates, commonly called costs of capital, are regarded as the minimum acceptable rates of return on invested capital which will maximize the economic value of ownership to the firm's existing shareholders. Outcomes of economic analyses are often quite sensitive to the interest rate used, and therefore it is important that the rate be carefully determined.

The procedure which should be followed for determination of a firm's cost of capital is a matter of sharp debate. The traditional viewpoint is that it should be some weighted average of the cost of the debt and equity funds used for investment. However, the weighting to use is still debated as is even the appropriate cost of the equity funds. For example, Bierman and Smidt (10) are of the school of thought which holds that equity funds cost should be based on the calculation of the expected dividends-price ratio, while Spencer and Siegelman (86) are of the school which holds that the cost of equity funds should be based on the expected earnings-price ratio.

Another idea on the determination of the cost of capital which has attracted much interest is based on an article by Modigliani and Miller (63). This viewpoint is that the average cost of capital to any firm is completely independent of its capital structure and is strictly the capitalization rate of future equity earnings.

A concept on the determination of the interest rate which seems most defensible is based on combined consideration of capital available and alternative projects available as well as the risk of individual projects. This viewpoint is appropriate for the usual situation in which there are desirable projects available requiring more capital than there is capital available for investment. Authoritative backing for this concept, which makes use of the opportunity cost principle, is discussed by Spencer and Siegelman (86-pp.403-404). Norman Barish (4-p.226) stated the concept well:

> The cost appropriate for establishing a minimum acceptable rate of return is essentially the opportunity cost for the money. The opportunity cost is the return which can be obtained, with comparable risks, by investing the firm's available funds either internally or externally.

It should be noted that this viewpoint on the interest rate refers to "available funds" and does not directly consider the source of the funds or the interest rate paid on those funds. It makes the assumption that there are other good projects available which would require more investment funds than there are funds that management is willing to make available for investment. This assumption is usually valid for firms that are at least moderately progressive in terms of improvement and expansion of facilities.

The Influence of Management on Project Outcome

In this research, only indirect account is taken of the fact that the ultimate outcome of a project is at least in part dependent on the interest and effort which management is willing to put into the project. Regarding the outcome of the project, Spencer and Siegelman say (86-p.372):

> Much will depend on factors outside of management's control--an economic recession, unexpected price competition in the industry, a sudden development of consumer resistance, the appearance of competing products serving the same end use--but a great deal will also depend on management's astuteness both in the original choice of the project from among the alternatives available and in their direction of it during its productive life.

In estimating element outcomes for a project, the analyst may either assume some fixed level of management effort and interest in the project, or he may try to consider the effect of varying management effort in estimates of the variation of element outcomes.

Scope of Study

This research will concentrate on the consideration of risk and uncertainty in two common types of capital economic analysis situations. The first type consists of analyses of mutually exclusive projects (alternatives) wherein a maximum of only one project can be selected out of a group. The second type of situation involves analyses of non-mutually exclusive projects wherein any number of projects can be accepted subject to limitations on individual project desirability and the total funds to be invested.

Four main elements (variables) will be considered in the quantitative developments. These are investment, life, salvage value, and net periodic (considered as annual) operating receipts or disbursements. The main quantitative results are comparisons of analysis outcomes when the risk is considered compared to outcomes when that risk is not considered. Extensive examination is made of the effect of variation of project life on the expected value of key factors in economic analyses. A model for approximating the expected value and variance of the outcome for a project when multiple elements of the project are subject to variation is presented and an example and discussion of its application is made. Recommended step-by-step procedures are developed for selecting among projects when variation of multiple elements is taken into account. Several disjoint extensions of methods for considering random variation of elements in capital investment analyses are developed.

Limitations of Study

The work herein generally assumes independence of analysis outcomes for the various projects considered. A notable exception is in Chapter V where covariance between project outcomes is considered in determining the variance of the distribution of difference between projects. However, no attempt is made in this work to consider the complementary and competitive effects of particular combinations of projects.

This research also does not take into account the effect of present actions on future choices. That is, it does not directly consider the effect that acceptance or rejection of particular projects will have on future projects available, the desirability of those future projects, and the capital available to be invested in the future.

No direct consideration is made of the effect of possible different income tax treatments for different alternatives. That is, such considerations as investment tax credits, initial or bonus depreciation allowances, differing tax rates, and tax on gain or loss on the disposal of the assets are not directly handled. All figures used in the analysis and comparisons made herein are assumed to be "after-tax" figures, i.e., figures after the effect of income taxes has been taken into account.

For the research on determining the effect on calculated results caused by the quantitative consideration of variation of elements, periodic net cash receipts or disbursements, D, are assumed to be uniform in amount over the life of the project. Also, it is assumed that the investment occurs in one lump sum at the beginning of the life of the project. While these may be stringent assumptions, they are made so that the range of conditions to be considered is more manageable in size.

Little attempt has been made to consider criteria except those which can be reduced to cost and revenue terms. However, it is recognized that any final decision between alternatives should be based on all relevant criteria, including intangibles, important to the decision. Also, no attempt has been made to consider the utility associated with different monetary outcomes; i.e., dollar value is taken to be equivalent to utility to the decision-maker for all projects considered.

The procedures, decision guides, and graphical analysis aids presented have not been tested for workability in any real situation. The workability of these developments depends not only upon their inherent soundness, but also upon the desire of the analyst and the willingness of management to use them effectively.

CHAPTER II

CONSIDERATIONS OF RISK AND UNCERTAINTY IN CAPITAL ECONOMIC ANALYSES--A REVIEW OF THE LITERATURE

Before discussing existing procedures for considering risk and uncertainty, it should be observed that these procedures fall far short of the general ideals of applicability and practicability, and that there is much room for development. The reader will recall the quotation by Roberts (75) in the Introduction. To further the point and make it more current, Hillier said in 1963 (48-p.443):

> Capital budgeting literature has not yet given much consideration to the analysis of risk; and such procedures as have been suggested for dealing with risk have tended to be either quite simplified or somewhat theoretical. Thus, these procedures have tended either to provide management with only a portion of the information required for a sound decision or they have assumed the availability of information which is almost impossible to obtain.

The typical prospective investment analysis does involve the estimation of elements which are subject to variation. When possible differences in alternatives under consideration are of sufficient consequence to the enterprise, it is worthwhile to explicitly consider the variation of those elements. As Hillier (48-p.443) says: "The amount of risk involved is often one of the important considerations in the evaluation of proposed investments."

In the following pages, approaches or procedures that have been described in the literature for making economic analyses in the face of risk and uncertainty will be discussed. In general, progression will be made from simple, common approaches through increasingly more sophisticated approaches; covering a heterogeneous group of specific, limited-use approaches as well as broad, general-use approaches.

Assumed Certainty

The most common approach presently used for making economic analyses is to "assume certainty," i.e., use only the mean expected values of the elements so as to get an "expected" result (measure of merit such as rate of return, present value, annual worth, etc.), and then to subjectively take into account, if at all, the effect of uncertainty or variation in those elements. This approach or procedure might best be called the "expectation" approach. This is the means traditionally advocated in engineering textbooks, and it has flourished in practice because it requires less estimating information and is less difficult to perform than other quantitative means.

Risk Discounting

A simplified approach for the consideration of risk and uncertainty which is often used in an economic analysis is to use an interest rate appropriate for the associated degree of risk as the standard for the minimum acceptable internal rate of return. Elaborations on this method can be found in Grant and Ireson (32-pp.143-144) and in Bierman and Smidt (10-Chap.9). This procedure is weak because it suppresses the information regarding the risk of the proposed investment project and fails to explicitly measure the risk of the investment. The approach is also weak because the specification of what interest rate is appropriate for what degree of risk is difficult and subjective, and because it fails whenever a project has a very short life such that the effect of discounting is not included in the analysis.

Conservative Adjusting

Another simplified procedure for considering risk and uncertainty which is analogous to and may be used in conjunction with risk discounting is to "adjust" all estimates in a conservative direction to reflect the uncertainty of those estimates. For example, estimated periodic receipts may be adjusted downward, estimated periodic disbursements may be adjusted upward, and/or estimated project life may be shortened. John McArthur (62) reflects efforts to gain accuracy when the practitioner is applying this approach to the consideration of dispersion of project life. He outlines the use of tables which show what "adjusted economic life" is appropriate for common patterns of skewness and modal values.

The conservative adjusting approach is weak, for it involves difficult and subjective estimates of how much adjustment should be made for varying degrees of risk or uncertainty. Further, in adjusting several parameters such as income, costs, and project life at once, one may well tend to be inconsistent and over-conservative in the final calculated analysis results.

Sensitivity Analysis

Another simplified approach for considering risk and uncertainty, which is commonly advocated, is often called "sensitivity analysis." Sensitivity analysis involves changing estimates of uncertain element(s)—(usually just one element at a time)—and investigating the sensitivity of the measure of the merit of the investment to such revisions in the element estimates. This procedure gives some indication of the effect if one of the original estimates is either too optimistic or too pessimistic. A short discussion of this procedure may be found in Chapter 13 of Grant and Ireson (32). Sensitivity analysis is quite limited in the amount of information it can provide, for it is difficult for one to draw precise conclusions about the possible effect of combinations of errors in the estimates. Many writers refer to sensitivity analysis as a common approach to exploring the effects of uncertainty, but most conclude that it has the pronounced limitations of lacking conciseness and comprehensiveness.

Expected Utility

A fairly-new, sophisticated approach for the consideration of uncertainty, which has received speculative and developmental attention in the literature, is what shall be called herein the "expected utility" approach. This procedure consists of determining the "utility," e.g., degree of usefulness or desirability to the decision-maker, of each of the possible outcomes of an investment and then determining the expected value or weighted average of the utility to use as a measure of the merit of the investment. Easy-to-read elaborations on this procedure may be found in Bierman, Fouraker, and Jaedicke (9-Chap.7) and Schlaifer (78-Chap.2). It may be well at this point to delve more deeply than the above references to explore the nature of the two somewhat controversial topics which underly this expected utility procedure--subjective probability and utility theory.

Subjective Probability

Subjective, or personalistic, probability statements are statements based on strength of belief, for they are not backed by data indicating actual frequency ratios, as are their opposites, objective probability statements.

In considering the risk or uncertainty in economic analyses, one is primarily concerned with the range of possible outcomes, and some "weighting of likelihood" or "probability" of the possible outcomes. Further, these analyses are normally of unique events which cannot be replicated. In so doing, one is concerned with subjective probabilities. Spencer and Siegelman say (86-p.16):

> Probabilities, therefore, should not be viewed for decision purposes as long-run frequency ratios since economic events rarely repeat themselves in a homogeneous manner. It is the degree of uncertainty that is relevant for decision making, and probability should be looked upon as a connecting link between the evidence available and the outcome being considered, with neither being necessarily measurable. As the available evidence becomes larger, the "weighted" probability of a particular event relative to others increases, and the degree of uncertainty thus diminishes accordingly.

Up until the 1950's, most statisticians rejected the idea of subjective probability, for the classical, inferential statistics of the school of Neyman and Pearson are based on the use of objective probabilities. One notable opponent of the use of subjective probability is the English economist, George Shackle (81) who rejects the idea of subjective probabilities for non-repetitive events, and substitutes the idea of "potential surprise."

Even today, some statisticians are uncomfortable with subjective probability. However, substantial philosophical justification by such leaders as Leonard Savage (77) and practical use by Schlaifer (78), Raiffa (74), and others has overcome much of the opposition so that subjective probability now seems to be generally recognized as a valid tool for situations in which the use of objective probability is not feasible or obtainable.

Examination will now be made of another topic, utility theory, which is intimately tied to the use of subjective probability in the application of the "expected utility" approach, and which is also important in other economic analysis approaches.

Utility Theory

As used in micro-economics, the concept of utility is psychologically oriented and refers to the subjective satisfaction derived by an individual from the possession of a given number of units of some commodity. The utility theory referred to herein is often thought of as a concept for measuring attitudes of an individual (decision-making unit) toward risk or uncertainty.

Daniel Bernoulli (8) is credited with publishing, in the eighteenth century, the first notable work on utility applied to risk situations. His ideas were mulled over by economists, mathematicians, and philosophers for two centuries to little avail. However, Von Neumann and Morgenstern (89), in their classic work of 1944, initiated among economists and statisticians an intense revival of interest in the concept of utility functions for individuals. Since then, limited applications and speculations on applications of individual utility functions have been steadily developing.

Once an individual's utility function and the subjective probabilities of all possible outcomes for a given alternative are obtained, the expected utility for that alternative can be calculated by summing the products of the utility of each outcome times the probability of each outcome over all possible outcomes.

The use of expected utility value as a decision criterion has a real advantage over the expected monetary approach. The expected monetary approach virtually overlooks the severe consequences of widely varying possible outcomes, and merely takes a weighted average of all outcomes. Expected utility value overcomes this objection by incorporating these variance influences directly into the computations. A large loss may be assigned a large negative utility by the individual, or he may assign a very great positive utility to a large increment in wealth, thus automatically bringing variance influences into the calculated results.

While there is a large amount of current literature introducing the expected utility value approach and speculating on its future usefulness, there seems to be a lack of demonstrated use up to this point. Hillier (48-p.445) examines the subject fairly closely, and concludes:

> Unfortunately, utility is a subtle concept, so that the measurement of utility is a difficult task. Therefore, it would be extremely difficult to determine explicitly, with all the needed precision, the utility to management of all the possible outcomes of an investment. From a practical point of view, management usually would have neither the time nor the inclination to participate in such a monumental task in a formal manner.

At this point, it is concluded that possible use of some utility approach to capital investment economic analyses is an interesting speculation. However, because of the difficulties of measuring individual utility functions, it seems that the potential of the expected utility approach, as presently developed, is not too great.

Probabilistic Monetary

An approach for dealing with risk and uncertainty in economic analyses of capital investments which is drawing extensive attention in the literature is named herein the "probabilistic monetary" approach. It involves constructing a model of the prospective investment or investments to be analyzed so as to reflect the variation of elements thought to be important in the analysis. If the model involves simple variation functions for parameters, then sometimes it can be mathematically analyzed so as to directly obtain the probability distribution of the measure of merit. If the variation functions and the interactions of parameters cannot be feasibly manipulated analytically, then the only practical way to obtain the probability function of the measure of merit is to use Monte Carlo simulation.

Bernhard (7-p.147), in his dissertation on *The Theory of Capital Investment Planning*, outlines some elaborate versions of this approach:

> A more elegant but less workable procedure recommended for handling uncertainty is to assume that for any possible combination of productive investments which may be undertaken, future net returns and interest rates may be described by a joint probability distribution. From these, a probability distribution of the present worth of each such combination may be derived.

Bernhard also discusses on pages 52 and 53 of (7) how the above procedure may be made even more elegant by considering sets of probability distributions of net returns corresponding to the set of alternative levels of output for each period and also by considering the correlations between different probability distributions within periods as well as between periods. While Bernhard's ideas on the probabilistic monetary approach are worthy, they have value primarily as concepts, for if application to normal investment planning problems are attempted, they may well require information that cannot be feasibly obtained and also calculations too voluminous to perform.

An outstanding work on analytical evaluations in capital investment analyses where elements vary was published by Hillier (48) in 1963. In this work, he requires that the cash flow each period be broken down into sub-flows, and that each sub-flow be estimated in terms of its expected mean, standard deviation, and degree of correlation from period to period. Given this information, his method allows one to generate an explicit description of the risk involved in terms of the probability distribution of the measure of merit. A description of the mathematical models developed is given in Chapter IV of this thesis.

Hillier (49) recently released an extension of the work in (48) in which he conceptually and mathematically shows how to optimize the overall capital budgeting problem through considering the interrelatedness of an entire set of investment proposals. This interrelatedness can consist of any degree of competitiveness or complementariness of any proposal with any other proposal. The essence of this approach by Hillier is to obtain estimates of the distribution of the periodic cash flow for each feasible combination of proposals, convert these to a distribution of present value of monetary cash flow for each combination of proposals; and then, if desired, convert these to an expected utility for each combination of proposals. His work is greatly enriched by various extended suggestions on how to implement the approach.

Hillier's formulation in (49), while being quite complete and general in concept, has the crippling drawback that it requires a prohibitive amount of work to study each of the multitudinous number of combinations of projects that comprise the usual capital budgeting situation. However, it should be acknowledged that he did develop some procedures and recommendations for considering subsets of proposals as a class and also for eliminating some proposals from extended study so as to cut down on the total amount of study required.

Joel Cord (16) recently published a new procedure for partially considering uncertainty in general investment analyses. He developed a mathematical method for optimally selecting investment projects out of a group with uncertain returns under conditions of limited funds and a constraint on the maximum average variance allowed in the investment projects selected. This is accomplished through introduction of the LaGrange multiplier and the reformulation of the problem as a recurrence relationship so that it can be solved by dynamic programming.

The concepts involved in Cord's analysis are somewhat related to Markowitz's work on portfolio selection (61), but there are several key differences which make Cord's work more general. First, Cord considers that an investment project can be either undertaken or not undertaken; there is no option to invest any amount on a continuous scale of possible amounts as in Markowitz's portfolio selection analysis. Secondly, Cord does not restrict consideration to quadratic objective functions as is done in Markowitz's portfolio selection analysis.

An outstanding recent article on the use of Monte Carlo simulation for probabilistic evaluation in investment analyses was written by

David B. Hertz (44). This article explains well the basics of the Monte Carlo approach and shows how it can be used to evaluate the effects of variation of elements so as to obtain more realistic answers--in terms of not only the expected outcome but also the range of outcomes together with associated probabilities.

Hertz points out that the main advantages of the probabilistic monetary approach using the Monte Carlo technique are: (a) the kind of uncertainty that is involved in the estimate of key factors is evaluated ahead of time so as to facilitate the fullest availability of information for decision-making, and (b) computer programs for Monte Carlo simulations can easily produce valuable information about the sensitivity of the possible outcomes and their associated probabilities to variation in input parameters. This article strikingly shows that consideration of probabilistic outcomes can be important. In an example problem given, the "assumed certainty" approach showed a rate of return of 25.2 per cent, while the "probabilistic monetary" approach showed an expected rate of return of only 14.6 per cent, together with explicit information on the variation of that rate of return.

Another good article on the Monte Carlo technique is by Hess and Quigley (46). It reflects the widespread developing interest in the Monte Carlo means of applying the probabilistic monetary evaluation approach for uncertainty in capital investment analyses.

Decision Trees

Another approach which adds realism to investment analyses is the use of "decision trees." A decision tree is a diagram which results from tracing out the future alternatives together with outcomes and associated probabilities which can result from a present decision. Quantification of the future outcomes and subjective probabilities and making the accompanying analysis enables one to make a present decision which includes consideration of the risk and effect of the future.

The idea of applying decision tree concepts to the analyses of sequential decisions over time seems to have good possibilities for future development. Magee (58)(59) has given several examples of important types of problems which can be studied by this approach. However, apparently they have not been used too much in practice, for they have only recently been advocated in literature such as the *Harvard Business Review* (58)(59). The main drawbacks of decision trees are the common difficulties of estimating future outcomes and associated probabilities and also the difficulties of handling situations where the outcomes and associated probabilities are not discrete.

Variable Discounting

J. Morley English (25) advocates taking into account increasing risk with more distant future by use of a varying rate-of-discounting such that the further one estimates in the future, the progressively higher will be the rate of discounting (and, thus, the lower will be the weighting of future outcomes relative to present outcomes). As an example of such a function, the interest, r, at any time in the future, t, could be said to vary according to the function

$$r = r_o e^{at} \tag{1}$$

where r_o is the initial (risk-free) rate and a (which is greater than zero) is the coefficient which determines the rate of increase. The advantage of this approach is that it provides a way to reconcile the short-run and long-run viewpoints.

The above function and other variable rate-of-discounting functions that English advocates in (25) are criticized by Hemmes (41) as not being readily useable. In (23), English acknowledges the deficiency and rather elaborately develops what he terms an "operationally useful discount function." That function expresses the interest rate as a function of time as

$$r(t) = \frac{1}{t} \log_e \frac{1}{1-r_o t} \qquad (2)$$

where all of the symbols are defined above and $r_o t$ is ≤ 1.

A great disadvantage of English's "operationally useful interest function" which should be noted is that its use fixes a planning horizon, T, which is the reciprocal of the initial, risk-free rate, r_o. That is, $T = 1/r_o$, and $r_o = 1/T$. This does not seem reasonable in many cases, for the existence of a low interest rate does not necessarily mean that there is a long planning horizon, and vice versa.

It should be noted that apparently none of English's approaches to considering uncertainty through special discount functions have attracted much attention, for there is a dearth of further development and integration of them in the literature.

Specialized Models

There are several models for considering risk and uncertainty in investment analyses that are specialized and of limited applicability, and hence will be covered only briefly. We will discuss these in turn below:

Freund (30), in 1956, developed a model which combines the use of a specific utility function for money, normal distribution of returns, and mathematical programming to choose an optimal portfolio of investments. It was a good forerunner, but has limitations similar to Farrar's (27) procedure (below).

Naslund and Whinston (68) developed a specialized programming model for investment in the stock market. Their approach involves the maximization of expected monetary gains subject to probabilistic constraints on maximum loss during the various periods. However, it does not appear that this approach can be extended to the general problem of investments under uncertainty.

Eisen and Leibowitz (20) showed conceptually how to handle uncertainty in a very limited class of problems in their article on "Replacement of Randomly Deteriorating Equipment." Their analysis requires a probability distribution of the capital replacement cost, given information on the extent to which the old equipment has deteriorated. They derive formulae for optimal replacement using two different decision procedures:

(a) replacement at a fixed age, and

(b) replacement when the cost-density reaches a given value.

It was shown that the use of decision procedure (b) leads to lower expected cost than decision procedure (a); however, they recognize that the cost of acquiring periodic current information may be so high as to more than offset the savings as compared to decision procedure (b).

John H. McArthur (62) demonstrated easy-to-calculate formulae for finding the present worth of a uniform series of cash flows for a project where the life of the project is distributed either according to the normal distribution or according to the gamma distribution. The only element subject to variation that McArthur considers is project life, but this is a key element in many economic analyses. This approach, unlike most of the approaches reviewed herein, is readily useable by the non-sophisticated practitioner.

Gordon M. Kaufman (51)(52) made noteworthy contributions with his formalized procedure for making sequential investment decisions under uncertainty. His developments are built upon mathematical formulations by James L. Fisher (29). Kaufman's work is developed to fit the repetitive drill-or-not-drill, buy-or-not-buy type decisions that oil and gas operations face.

He presents eight different models which contain different combinations of very restrictive assumptions pertaining to the investment required, the payoff to be obtained and the number of potential investments to be available. For example, the models assume that the amount of investment required for individual opportunities is either uniform or known with certainty or conditional probabilities are known; and that the number of opportunities in a period is either known or varies with a known average.

There are several weaknesses of Kaufman's developments which are related to his restrictive assumptions. For example, he considers only one investment at a time, not the more general case where one knows about several investments at a time and wants to choose from the group. Also, he assumes that the investments are independent and not conditional such that present decisions can affect future alternatives.

In summary, Kaufman's developments are unique contributions because they provide a formal mechanism for taking into account the investor's future expectations about investment opportunities. However, his mathematical formulations are for situations bounded by such stringent assumptions that their applicability is quite limited. There is much room for extending his work. However, it seems that the level of mathematics that would be required for analysis when some of his assumptions are altered for greater generality would be prohibitive.

Expectation-Variance

An approach for the consideration of risk and uncertainty which has been the topic of considerable recent research is commonly referred to as the "expectation-variance" or the "certainty equivalence" approach. In general, the certainty equivalent of an investment, v, is said to be a function of the expected value of the alternative, possible outcomes, u, and the variation of that outcome, σ. That is, $v = f(u,\sigma)$. While the nature of the function outlined above is open to question in most cases, there is general agreement that increases in an investment's expected value, u, tend to increase the desirability of the investment, while increases in the variation tend to decrease the desirability of the investment. One simple example of this model is:

$$v = u - A\sigma \tag{3}$$

where A is commonly referred to as a "coefficient of risk aversion." Farrar (27-p.15) says that the certainty equivalence approach originally failed to attract many followers. He backs this statement up with reasons, the most important being that it attempts to examine each asset as an individual, and not as a portion of a larger set of holdings.

Markowitz (61) developed a means which allows the certainty equivalence approach to overcome this disadvantage. His development is basically a mathematical programming model and it includes the consideration of covariance between the measure of merit of a given investment and that of alternative competitive or complementary investments. Markowitz's approach enables one to select efficient portfolios from the set of all possible portfolios. There are several severe drawbacks to the use of his method in general capital investment analyses. His method is designed for use in conditions where one has the flexibility and availability of information that is characteristic of stock portfolios, and not general projects for investment. That is, he assumes the availability of information on the mean and variance of the return for each security, as well as the covariance of return for each pair of securities. Further, he assumes that investments in each alternative can be made in continuous (non-discrete) amounts and that there are no mutually exclusive alternatives (where the choice of one precludes the choice of any other of a group). Last, but not least, his procedure requires explicit specification of the decision-maker's coefficient of risk aversion. Hence, Markowitz's procedure is not directly applicable to capital investment analyses of projects. However, it has served as a point of departure for further developments.

Donald Farrar (27) combines and extends aspects of certainty equivalence, classical utility theory, and Freund's programming model to create a specialized version of the certainty equivalence procedure. Farrar's model has several advantages, but it also has several stringent limitations. It can represent only a quadratic approximation to an investor's utility of money schedule, and it is designed for use where one has the availability of information and flexibility that is characteristic only of financial portfolios (stocks and bonds). Also, the model does not provide an objective criterion according to which one can evaluate the coefficient of risk aversion.

As an explanatory digression, Farrar's model, which is a mathematical programming model, can be written mathematically as:

maximize:

$$U = \sum_i \gamma_i \mu_i - A \sum_{ij}\sum \gamma_i \sigma_{ij} \gamma_j \tag{4}$$

subject to:

$$\sum_i \gamma_i = 1, \text{ and}$$

$$\gamma_i \geq 0 \text{ for all } i;$$

where U = Utility of portfolio of projects,

γ_i = Proportion of portfolio which consists of the i^{th} project,

μ_i = Mean return on the i^{th} project,

σ_{ij} = Covariance of returns of i^{th} and j^{th} projects, and

A = Coefficient of risk aversion.

After recognizing the stringent limitations of his model as discussed above, Farrar enters a note of optimism (27-p.78) on the possibilities of future developments to make his work more useful:

> Just how far it [his model] can be pushed outside the financial realm is, of course, unclear. Indeed, it is probable that with additional constraints, its applicability as a model of the investment decision under uncertainty may be quite general.

Cramer and Smith (17) recently published substantial developments toward extending Farrar's model to general investment decisions. They use classical procedures in a special way for determining utility functions so as to obtain indifference points for expected monetary value versus standard deviation of monetary value. From plots of these points, they show how to determine all constants necessary to express certainty equivalence models in a form more complex than have previously appeared. Cramer and Smith view certainty equivalence as being analogous to utility, and their model appears in the form:

$$U = u - A\sigma^a I^b \tag{5}$$

where U = utility or certainty equivalence,

u = expected monetary outcome,

A = coefficient of risk aversion,

σ = standard deviation of monetary returns,

I = investment amount, and

a and b = constants.

Cramer and Smith's article is of particular value because it shows the results of an actual study which was made to test the workability of their conceptual contributions.

J. Morley English (25) originally advocated in 1961 that the coefficient of risk aversion, A, be subjectively determined as a function of the amount of investment required for each individual alternative. He maintained that the effect that larger capital committments have on increasing the variance of outcomes should be reflected in an increasing coefficient of risk aversion. The effect of size of investment is taken into account directly in the certainty equivalence model of Cramer and Smith shown in Equation (5).

English and Haase (26) have done substantial pioneering extensions to the expectation-variance approach to make it more useable. They point out, paraphrasing Morris (65-pp.215-216) and Farrar (27-p.26),[2] that the choice of coefficient of risk aversion, A, represents a measure of the utility associated with the confidence with which an investment is made.

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2. Farrar further points out that as long as there is a diminishing marginal utility of money, the correspondence between a firm's coefficient of risk aversion and the firm's utility function curvature is $A = (-U''(u))/2$, where u is the mean monetary outcome. Expressed in words, the coefficient of risk aversion is equal to the negative of 1/2 of the second derivative of the utility of money function evaluated at the mean monetary outcome.

English and Haase further postulate that a choice of A is equivalent to making the following probability statement concerning the present worth of an investment:

A is selected so as to satisfy

$$\text{Probability } [A(T) \geq \{E(A(T)) - A\sigma_A\}] = \upsilon \qquad (6)$$

where $A(T)$ = present worth of investment,

$E(A(T))$ = expected present worth of investment,

A = coefficient of risk aversion,

σ_A = standard deviation of present worth of investment, and

υ = probability level.

However, English and Haase maintain that there often is not satisfactory intuitive basis for making such a probability assessment. To meet the deficiency, they develop a procedure for estimating and making probability statements about future cash flows and formulae for working backwards to find A. In addition, they develop a new criterion which is parallel to the expectation-variance criterion but which is based on the probability assessments of cash flows. The authors do not attempt to name this new criterion, but it might well be named the "expectation-variance of individual cash-flow" criterion. The justification for this new criterion is that it forces judgment (intuition) to go back as far as possible toward the elemental factors (individual cash flows) that may more readily be assessed intuitively, thus allowing for more reliable assessment of risk.

Statistical Decision Theory

A rather sophisticated approach which has future possibilities but which is relatively underdeveloped in application to investment analyses is the use of the concepts and techniques of statistical decision theory. In the sections which follow, statistical decision theory will be examined concerning its nature and characteristics, development and literature, and potential application to investment analyses.

Nature and Characteristics

Statistical decision theory is an outgrowth of classical inferential statistics, but it has never been restrictively, clearly defined so that there is common acceptance of its definition. Hence, the viewpoints of several important writers in the field will be reflected below.

A lucid, though broad, characterization was given by William Morris (65-p.371): "A statistical decision differs from ordinary decisions only in that the process of collecting the data, drawing the inferences from it, and determining the decision, are included in a single body of theory."

The old, classical procedure of inferential statistics in hypothesis testing (using the Neyman-Pearson Rule) may be paraphrased as: For a given probability of a Type I error, select the test which minimizes the probability of a Type II error. In contrast, the statistical decision approach involves considering the costs of Type I and Type II errors as well as the cost of sampling against the reduction in error costs which will result from taking larger samples or additional samples.

Raiffa and Schlaifer (74-p.vii) view statistical decision theory as " . . . the mathematical analysis of decision making when the state of the world is uncertain but further information about it can be obtained by experimentation." They further characterize statistical decision theory as using numerical utilities to express the decision maker's preferences for consequences and also subjective probabilities to express the decision maker's degree of belief in the possible states of the world.

In total, probably the outstanding characteristic of statistical decision theory is that it provides a basic integration of the processes of information collection and decision.

Development and Literature

Abraham Wald (90-p.28), the foremost pioneer of statistical decision theory, in 1950 said of the early history:

> Until about ten years ago, the available statistical theories, except for a few scattered results, were restricted in two important directions: (1) Only decision functions were treated for which experimentation is carried out in a single stage; (2) The decision problems were restricted to problems of testing a hypothesis, and that of point and interval estimation.

A major advance in the theory of statistical experimentation took place during World War II with the development of sequential analysis. Sequential analysis is a method of statistical inference in which the number of observations and/or actions to be taken depends on the results of previous experimentation. Wald did considerable work on the general theory of non-sequential and sequential decision functions, culminating in his classic work of 1950 which was cited above.

Wald credits Stein (87) as being the first to formulate a model for statistical decision procedures which includes the design of experimentation (selection of the chance variables to be observed) as part of the decision problem. However, general credit is given to Wald for the real development of the decision theoretic approach; i.e., the formulation of statistics as decision making under conditions of risk.

Weinwurm (92) credits Schlaifer (78) for making a major contribution to statistical decision theory by replacing Wald's objective probabilities with the concept of subjective probabilities. Savage (77) was the leader in the early philosophical justification for subjective probabilities, but it was Schlaifer who pioneered in the popularization of how to use them in practical statistical decision problems.

A technique often associated with statistical decision theory is game theory. The theory of games involves decisions where there is conflict or competition as well as uncertainty as to the strategies of one's adversary(ies), whether that adversary be another person or a malevolent nature. Game theory has attracted much attention since it was expanded in the definitive book by VonNeumann and Morgenstern (89) in 1944. Blackwell and Gershick (11) in 1954 published a notable work which attempted to integrate games and decision theory. This was followed by similar work by Luce and Raiffa (55) in 1957. In the past decade, there have been numerous other works on both game theory and decision theory. Morris (67-p.379) says of the integration of game and decision theory:

> Within the context of statistical decision theory it has been customary to suggest that the decision should be viewed as a game. That is, one might imagine the possible futures as being the pure strategies of nature. The decision maker then assumes that it is important to protect himself as best he can against the worst possible "play" by nature, where nature is thought of as having the intelligence and aims of an opponent in the game theory sense.

Despite the popular interest in game theory over the past two decades, it still has severe limitations. The difficulty with game theory applied to statistical decision problems is that for problems of practical size and complexity, means of making computations either have not been developed or are exceedingly formidable. Computational procedures for competitive games are readily available for simple, limited problems like two-person-zero-sum games; but are not, in the present state of the art, reasonably practical for situations like multiple-person, non-zero-sum games. Nevertheless, the philosophy of game theory for analyzing conflict or competition and making explicit decisions in a statistical situation has been identified by some writers as an important contribution to statistical decision theory.

A practice that is commonly associated with statistical decision theory is the use of what is called Bayesian statistics. This practice is characterized primarily by frequent use of Bayes' theorem (see 57-p.161) and (14-p.176) to adjust "a priori" subjective probabilities for an unknown parameter to more-reliable "a posteriori" probabilities based on the results of sample evidence. There have been many treatises in recent years on uses of Bayes' theorem. Schlaifer (78), in 1959, published the first popular-consumption text which used these concepts.

Many present-day writers use terms like "Bayes decision rule," "Bayes solution," "Bayes strategy," etc., to mean the course of action which optimizes expected outcome (monetary, utility, risk, or whatever the measure of merit) where subjective probabilities are used to assign weights to the different possible outcomes and the true outcome is a random variable rather than a constant. The Bayesian approach to statistical decisions is described very lucidly by Jack Hirschleifer (42). It is thought by many to have merit for outstanding impact in future use. Chapter VIII contains examples of the use of Bayesian statistics for revising subjective estimates for economic analyses.

The most substantial recent work in this field is *Statistical Decision Theory*, published in 1961 by Lionel Weiss (93). Its main contributions are that it extensively examines the construction and use of Bayes decision rules, it covers the application of linear programming to satistical decision problems involving the minimax decision rule, and it summarizes Wald's work on problems involving a sequence of decisions over time.

An excellent work for ease of reading and comprehensive insight into the subject is *Elementary Decision Theory*, by Chernoff and Moses (14). Morris (67) effectively devoted several chapters to elements of statistical decision theory. Finally, Bierman, Fouraker, and Jaedicke (9) did an excellent job of presenting applications of some of the concepts of statistical decision theory in a somewhat over-simplified, yet very easy-to-understand fashion.

Application to Investment Analyses Under Risk and Uncertainty

There are several drawbacks to the presently-existing statistical decision theory which seem to limit its applicability in analyses of the effect of risk and uncertainty in economic evaluations of capital investments.

The statistical decision theory approach normally involves taking samples (observations of relevant variables) so as to arrive at "a posteriori" probabilities, combining these with loss or gain functions for each possible strategy (alternative), and calculating the optimal strategy. Analyses for investment decisions, on the other hand, by nature do not involve sampling in the usual sense of physically sampling outcomes to obtain information on the state of nature, for there is not normally a parent population from which to sample. However, they do involve estimations which are based, at least in part, on past data.

Another drawback of the presently-existing statistical decision theory is that the states of nature are normally limited to a rather small number of discrete possible outcomes for computational practicability. In most capital investment analyses of practical importance, these possible outcomes are either non-discrete or of great number, making the needed computations quite difficult.

One concept arising as part of statistical decision theory which seems to have potential for important use in investment analyses is that related to answering the question of how much cost of experimentation or further study is justified in order to gain certain reductions in uncertainty in the decision problem. Schlaifer's (78-Chapters 7 and 30) development of the rationale for calculating the expected opportunity loss (cost of uncertainty) has provided an approach for answering this question. This is shown in Chapter VII in the section on the valuation of decreased lack of certainty.

Miscellaneous Decision Rules for Complete Uncertainty

In this section, some arbitrary decision rules or principles for choosing between alternatives in situations where there is the element of complete uncertainty about certain probabilities will be discussed. These decision rules apply to situations where there is a number of alternatives (courses of action) and a number of possible outcomes (states of nature) and where the effect of each alternative on each possible outcome is known but the probability of occurrence of each possible outcome is not known.

Maximin or Minimax Rule

The maximin rule suggests that the decision maker examine the minimum profit associated with each alternative and then select the alternative which maximizes the minimum profit. Similarly in the case of costs, the minimax rule suggests that the decision-maker examine the maximum cost associated with each alternative, and then select the alternative which minimizes the maximum cost. These decision rules are conservative and pessimistic, for they direct attention to the worst outcome and then make the worst outcome as desirable as possible. However, they are widely discussed and form the usual basis for game theory analysis.

Maximax or Minimin Rule

These rules are direct opposites of their counterparts discussed above, and thus reflect extreme optimism. The maximax rule suggests the choice of the alternative which will maximize the maximum profit for each possible outcome, whereas the minimin rule suggests the choice of the alternative which will minimize the minimum cost for each possible outcome.

Laplace Principle or Rule

This rule simply assumes that all possible outcomes are equally likely and that one can choose based on expected outcomes as calculated using equal probabilities for all outcomes. Morris (66-p.313) says that there is a common tendency toward this assumption in situations where there is no evidence to the contrary, but that the assumption (and, hence, the rule) is of highly questionable merit.

Hurwicz Principle or Rule

This rule is intended to reflect any degree of moderation between extreme optimism and extreme pessimism which the decision maker may wish to choose. The rule may be stated explicitly as (65-p.314):

> Select an index of optimism, a, such that $0 \leq a \leq 1$. For each alternative, compute the weighted outcome: (a)•(Value of profit or cost if most favorable outcome occurs) + (1-a)• (Value of profit or cost if least favorable outcome occurs). Choose the alternative which optimizes the weighted outcome.

A practical difficulty of the Hurcwicz rule is that it is difficult for the decision-maker to decide on the value of a, the weighting factor. The Hurwicz rule also fails to have several of the desirable properties of a good decision rule, and can even lead to results which are obviously counter to one's intuition.

Minimax Regret Rule

This rule is similar to the minimax and maximin rules, but is intended to counter some of the ultra-conservative results given by those rules. This rule suggests that the decision maker examine the maximum possible regret (loss because of not having chosen the best alternative for each possible outcome) associated with each alternative, and then select the alternative which minimizes the maximum regret.

Summary

The choice of a decision rule or principle such as one of the above is often rather arbitrarily based on taste, intuition, and judgment of appropriateness for a particular situation. The greatest defense for the use of any of these principles is that their use will promote explicitness and consistency in decision making under complete uncertainty.

Examination of the literature and discussion of many different existing or potential approaches for considering risk and uncertainty in economic analyses is now complete. The next chapter will begin the presentation of research results.

CHAPTER III

KEY FACTORS IN ECONOMIC ANALYSES WHEN PROJECT LIFE IS A RANDOM VARIABLE

Introduction

In this chapter, the effect of risk concerning one element in economic analyses will be considered. This extremely important element is the life of a project. A project is defined here as an investment alternative which is separable for economic analysis purposes. The economic life of a project may be loosely defined as the length of time that a project is most economical in serving its intended purpose. The life of a project is just the length of time that the project remains in service, which, hopefully, is the economic life.

In making this rather detailed study of the effect of variation of the life of a project on key factors in economic analyses, certainty will be assumed for the other elements in the analysis so as to allow concentration on rather thorough exploration of the effects of dispersion of this one element. The work of this chapter will be based on assumed known distributions of life. The effect of different degrees of uncertainty as to what distribution is applicable will be implicitly considered by comparison of the results for different distribution types and amounts of variance.

Key Factors Considered and the Expectation Approach

The first key factor to be considered is the capital recovery factor, which is used in determining the equivalent uniform annual cost of depreciation plus interest for a project. This will be followed by an examination of the relation of the capital recovery factor to the capital recovery cost for various salvage value functions. The other key factor to be considered is the present value factor, which is used in determining the equivalent worth at some base time of a series of periodic receipts or disbursements.

The effect of different distributions of life will be taken into account by the calculation of the expected values of key factors and the comparison of these expected values with corresponding key factors assuming certainty at the expected life. The key factor using assumed certainty at the expected life serves as a basis for comparison because the expected life or a life close to the expected life is often used as the life in assumed certainty studies.

Another basis of comparison for the expected values of key factors is with the same key factors using assumed certainty at the modal life. The key factor at the modal life serves as a possible useful basis for comparison because the modal life may be used as the estimated project life in assumed certainty studies. That is, the practitioner may consciously or unconsciously choose as the assumed certainty life the life which has greatest probability of occurrence rather than the true expected, or mean, life. Thus, it may be of interest to know how the expected value of a key factor compares with that factor based on the modal life.

The use of the expected value of a key factor in an economic analysis for risk and uncertainty problems has both strong justifications and, in certain situations, definite weaknesses. William Morris (65-pp.210-211) says of the justifications:

> By far the most-used principle of choice in risk-taking situation calls for selecting the alternative which minimizes expected loss or maximizes expected gain. . . . Based on the "weak law of large numbers" and the "law of long run success" one can say that no other principle of choice will be as good as the expectation maximizing principle in the long run. . . . Even if we are not faced with a large number of repetitive decisions, the firm should apply the principle to many different decisions under risk and thus realize the long run effects.

The strongest weakness of the expectation principle is that the expectation has empirical meaning only if a very large number of similar decisions are involved. If only one or a small number of similar decisions are involved, as is usually the case in economic analyses, the actual outcome or average of outcomes may differ widely from the expected outcome.

It is widely recognized that the expectation principle is of little use in unique decisions which are extremely critical to the well-being or even existence of the enterprise. In such cases, the effect of unfavorable outcomes must receive more weighting than to merely serve to make the expected measure of merit less desirable according to arithmetic averaging. Morris recognized this in the following statement (65-p.211): " . . . there may still remain a very small number of highly crucial decisions which perhaps involve the very survival of the firm. For these few expectation maximizing may still be unsatisfactory."

Life Distributions Considered

The life distribution of a project may or may not be estimated with a high degree of confidence, depending upon circumstances, particularly the availability of pertinent mortality data. In order to estimate the life distribution with a high degree of confidence, mortality data must come from past projects under conditions that are essentially identical to those expected for the project(s) in question. If such data exist, the life distribution may be viewed as an objective probability distribution. Seldom, however, are such past mortality data available. In such cases, the distribution of life of a project must be estimated from past experience with projects that are more or less similar, or from considered judgments.

The number of different life distributions that could be appropriate for projects in general is unlimited. These distributions might be symmetrical, left skewed (mode on right), or right skewed (mode on left), with almost any given shape, mean, and variance. Hence, it seems reasonable to consider the effect of a wide range of life distributions in this study.

The most extensive work on determining what life distributions are appropriate for typical industrial property was reported by Robley Winfrey in 1936 (96). In Winfrey's work, 18 distributions were selected as being sufficiently representative of industrial property retirement experience. These distributions have come to be known as the "Iowa 18 type curves." They consist of a group of six symmetrical curves, a group of seven left modal curves, and a group of five right modal curves.

In this analysis, commonly known distribution functions which seem to be good representations of possible project life distributions will be considered. The term "commonly known" is used for distributions that are described in most applied statistics texts. Such distributions appropriate to this study are the normal, uniform, gamma, and beta distributions. Another distribution considered in this study which is not necessarily commonly known is the general triangular distribution. The general triangular distribution has the advantages that it can be easily specified and used by the practitioner and that it can be either symmetrical, left skewed, or right skewed to almost any degree.

The distributions considered in this study fairly closely overlap and approximate all the Iowa 18 type curves except for the extreme left modal curve, the extreme right modal curve, and the most peaked (and least variance) curve of each of the three groups. Since there is very little effect on expectations due to dispersion according to the most peaked curves and since the life distributions are usually subjective, it is felt that the commonly-known distributions considered herein form a good basis for study of the effect of different life distributions. Further, these distributions can be used readily by the practitioner wishing to study the effect of life dispersion for any condition which conforms to the shape of one of these distributions, but which involves a mean and variance combination not covered in this study.

The purely symmetrical distributions considered are the normal and the uniform. The gamma distribution is considered as a distribution with various shapes and a right skew. The beta and triangular distributions are considered as either symmetrical, right modal, or left modal shaped distributions. Figures 1-4 show graphically the shape of each of the distributions considered where coefficient of variation of life is expressed as a per cent. The particular distribution parameters and associated coefficients of variation which are studied extensively are shown in these graphs. For cases where the meaning of the distribution parameters is not clear, explanation is given on the figures for the distribution. The term spread is used to mean (Z-A)/E(T). Below is outlined the mathematical formulas of the density functions for each of these distributions.

Let T = life of project (in years),

$E(T)$ = expected, mean, or average life,

Z = maximum possible life, and

A = minimum possible life.

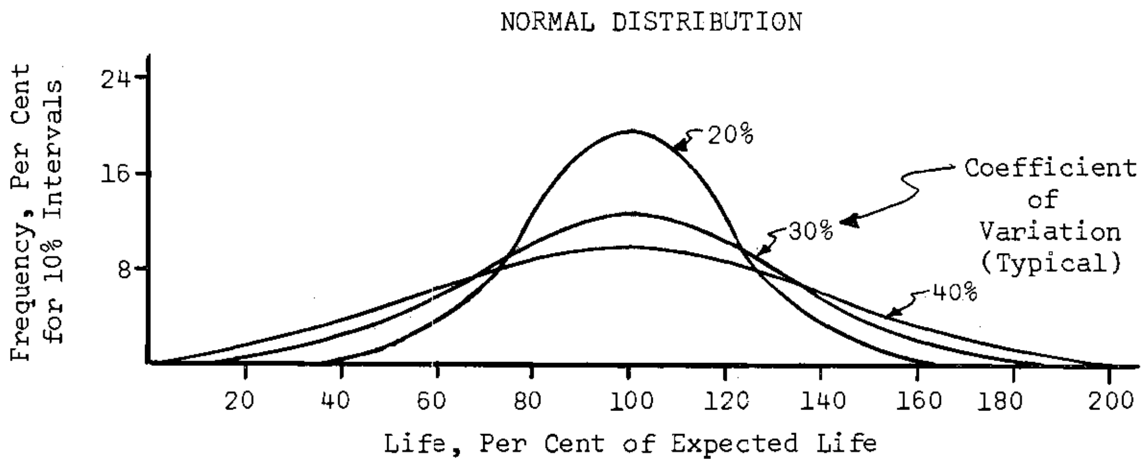
(a) Normal distribution:

$$f(T) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\left[\frac{(T-E(T))^2}{2\sigma^2}\right]}; \quad -\infty \le T \le \infty,^{2} \tag{8}$$

where σ = standard deviation of T.

(b) Uniform distribution:

2. Note: As a practical matter, T must be ≥ 0. In this research, σ is small compared to $E(T)$ so that the theoretical probability that $t < 0$ is small enough that there is negligible effect on calculated results due to this disparity.



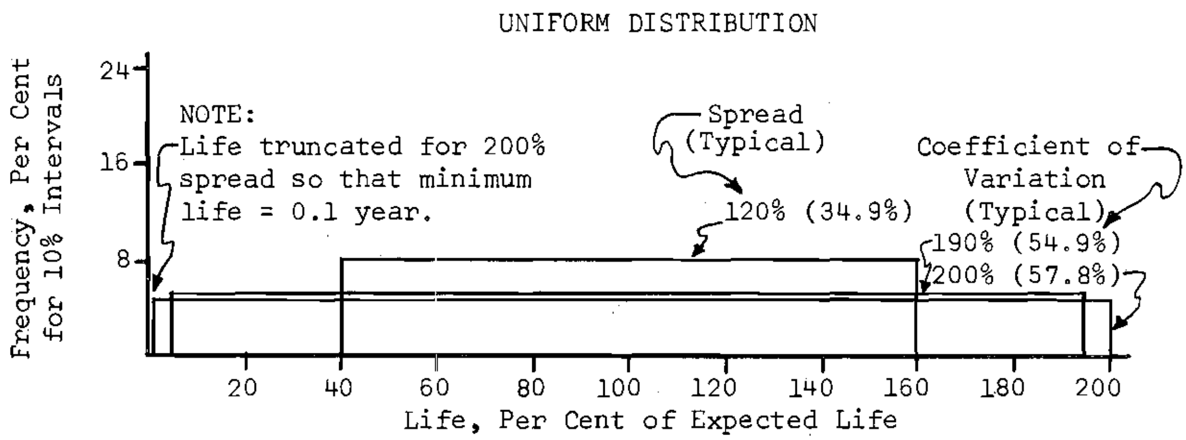


Figure 1. Distributions Considered

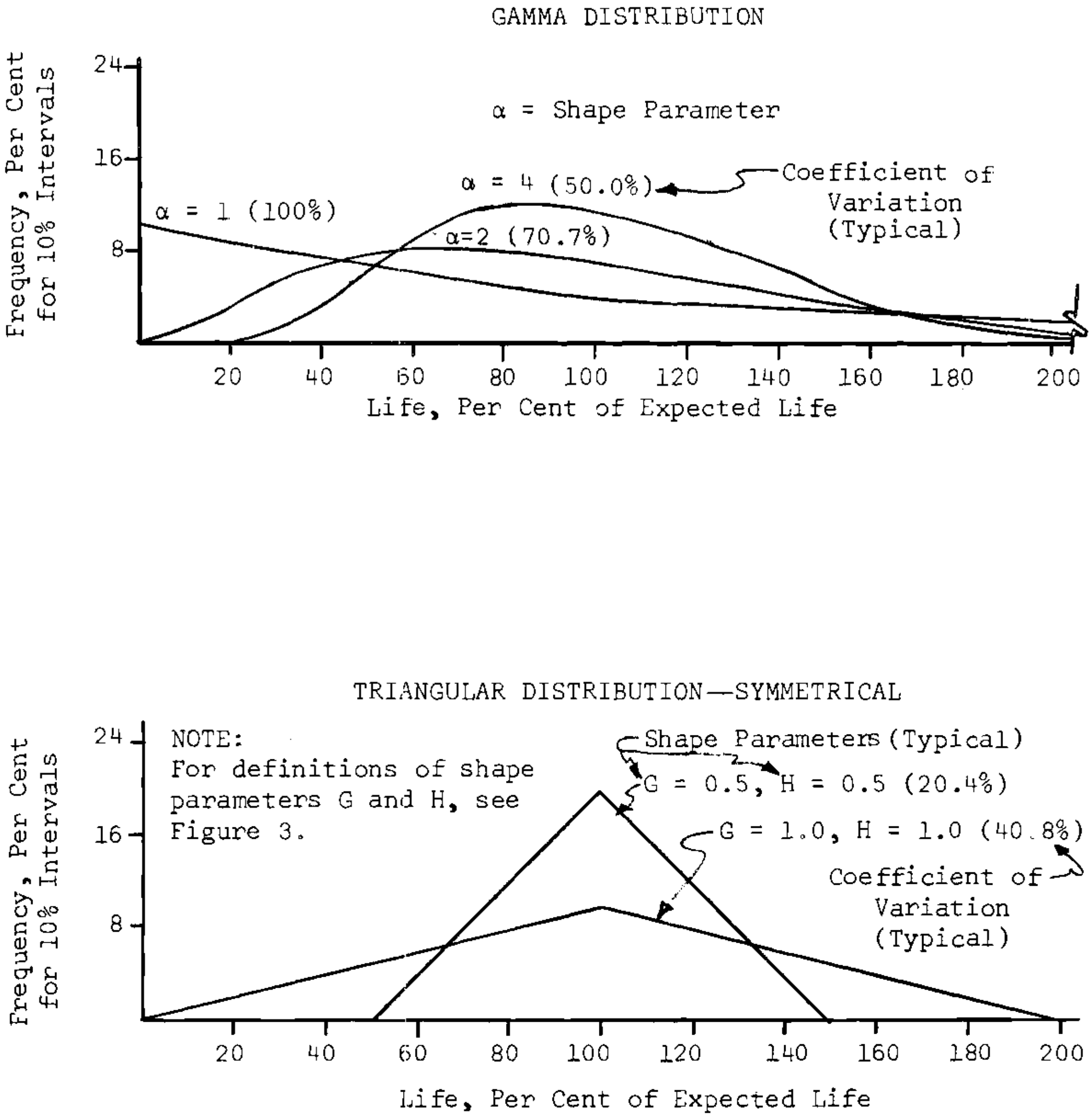
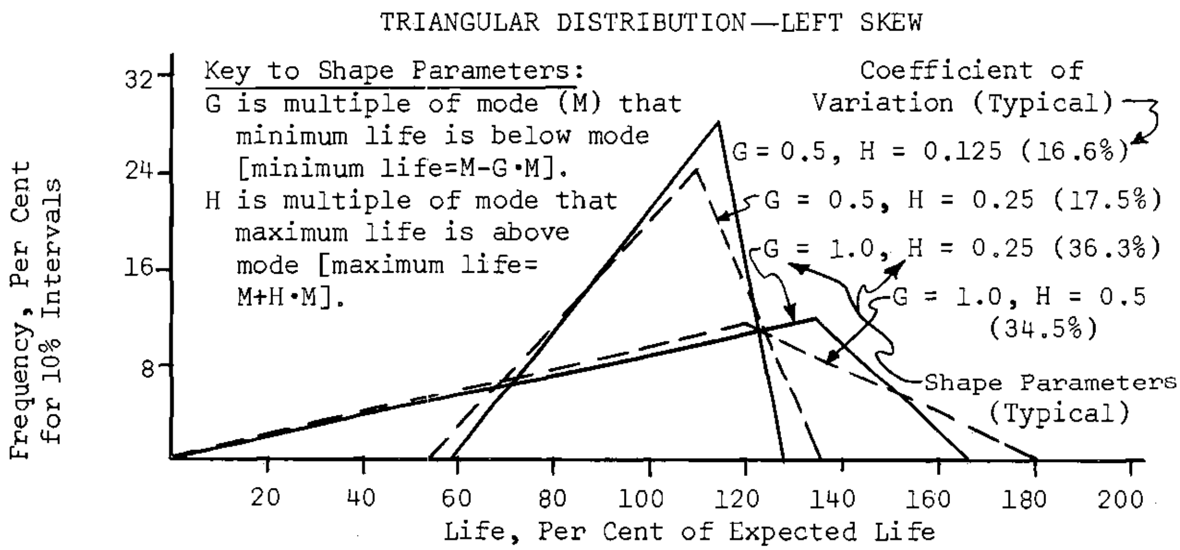


Figure 2. Distributions Considered



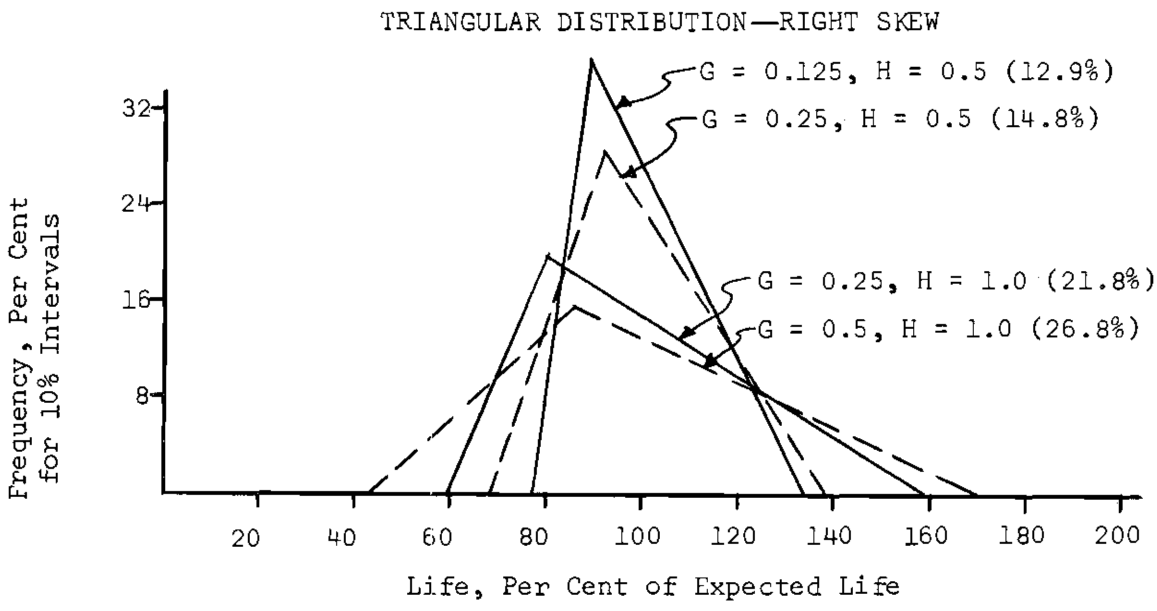


Figure 3. Distributions Considered

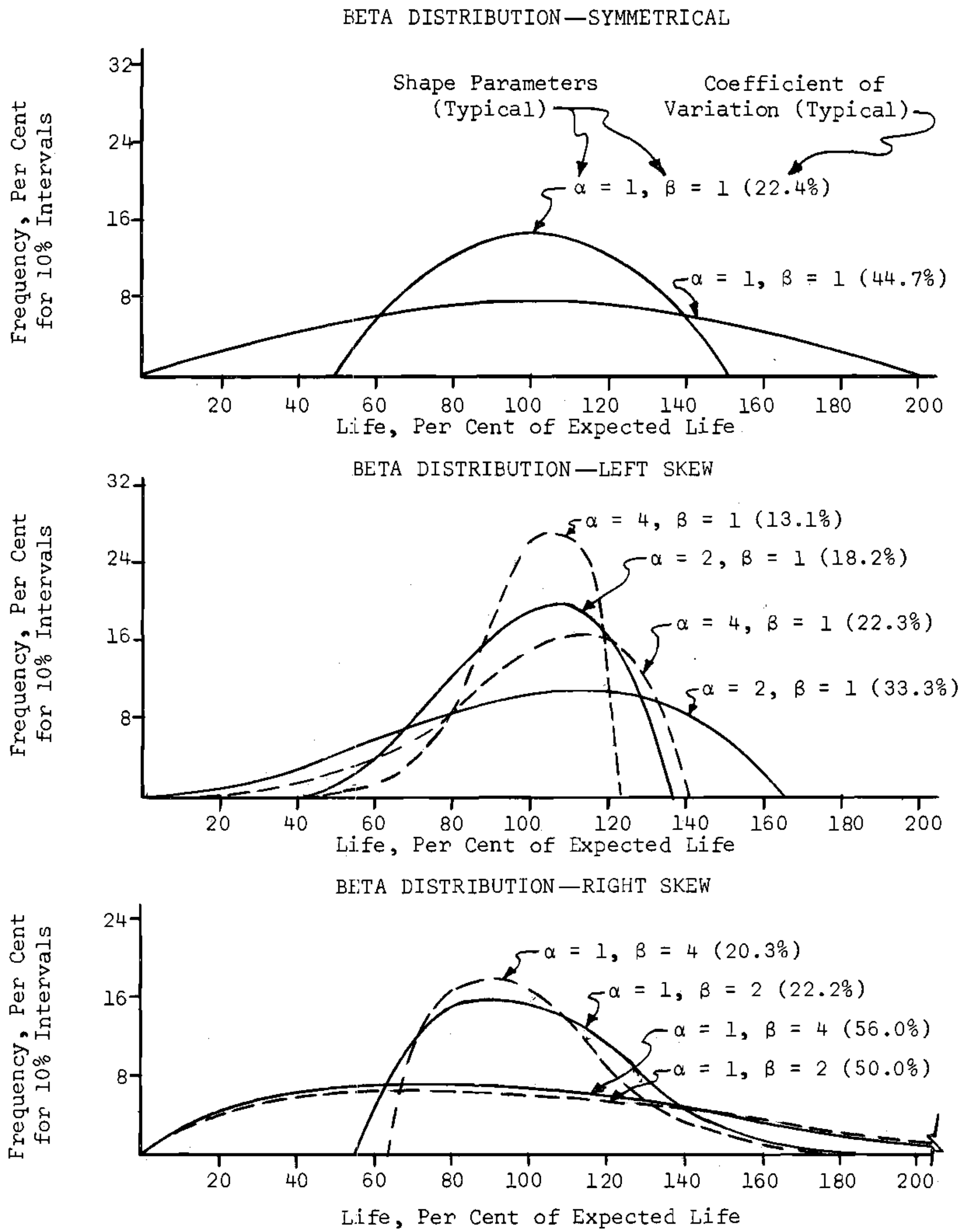


Figure 4. Distributions Considered

$$f(T) = \begin{cases} \frac{1}{Z-A}; & A \le T \le Z \\ 0; \text{ otherwise.} \end{cases} \quad (9)$$

(c) Gamma distribution:

$$f(T) = \begin{cases} \frac{1}{(\alpha-1)!\ \beta^{\alpha}} \cdot e^{-T/\beta} T^{\alpha-1}; & 0 \le T \le \infty \\ 0; \text{ otherwise.} \end{cases} \quad (10)$$

where α = shape parameter, and

β = horizontal scale parameter such that mean life, $E(T) = \alpha\beta$.

(d) Beta distribution:[3]

$$f(T) = \begin{cases} (\frac{1}{Z-A}) \frac{(\alpha+\beta+1)!}{\alpha!\beta!} (\frac{T-A}{Z-A})^{\alpha} (\frac{Z-T}{Z-A})^{\beta}; & A \le T \le Z, \\ 0; \text{ otherwise,} \end{cases} \quad (11)$$

where α and β are shape parameters.

(e) Triangular distribution:[4]

$$f(T) = \begin{cases} 2(T-A)/[(Z-A)(M-A)]; & A \le T \le M \\ 2(Z-T)/[(Z-A)(Z-M)]; & M \le T \le Z \\ 0; \text{ otherwise,} \end{cases} \quad (12)$$

---

3. See Appendix C for sketch of derivation.

4. See Appendix D for sketch of derivation.

where M = mode of distribution.

For the studies of the effects of different life distributions on key factors, graphs are shown to relate effects for expected lives over a continuous range from 2 to 30 years. Interest rates considered for all cases were 5 per cent, 10 per cent, and 20 per cent.

Effect of Various Life Distributions on Expected Capital Recovery Factor

Figure 5 shows the behavior of the capital recovery factor, $i/(1-e^{-iT})$, for a range of lives and interest rates. Note that the factor is quite negatively sloped for low lives and becomes almost constant for sufficiently high lives. The expected capital recovery factor, E(CRF), is calculated by the relation:

$$E(CRF) = \int_A^Z i/(1-e^{-iT}) \cdot f(T)dT \qquad (13)$$

where $i/(1-e^{-iT})$ is the capital recovery factor for the continuous compounding, continuous flow case, and all other symbols have previously been defined.

The main intent of this study is to produce quantitative information which will help the practitioner decide when it is worthwhile to quantitatively consider dispersion of life through calculating E(CRF) rather than calculating CRF based on the expected life (denoted "CRF at Expected Life") or even based on the modal life (denoted "CRF at Modal Life"). The main bases for comparison are simple ratios. The ratio of greatest interest is R_{cr} = E(CRF)/CRF at Expected Life. Another ratio of

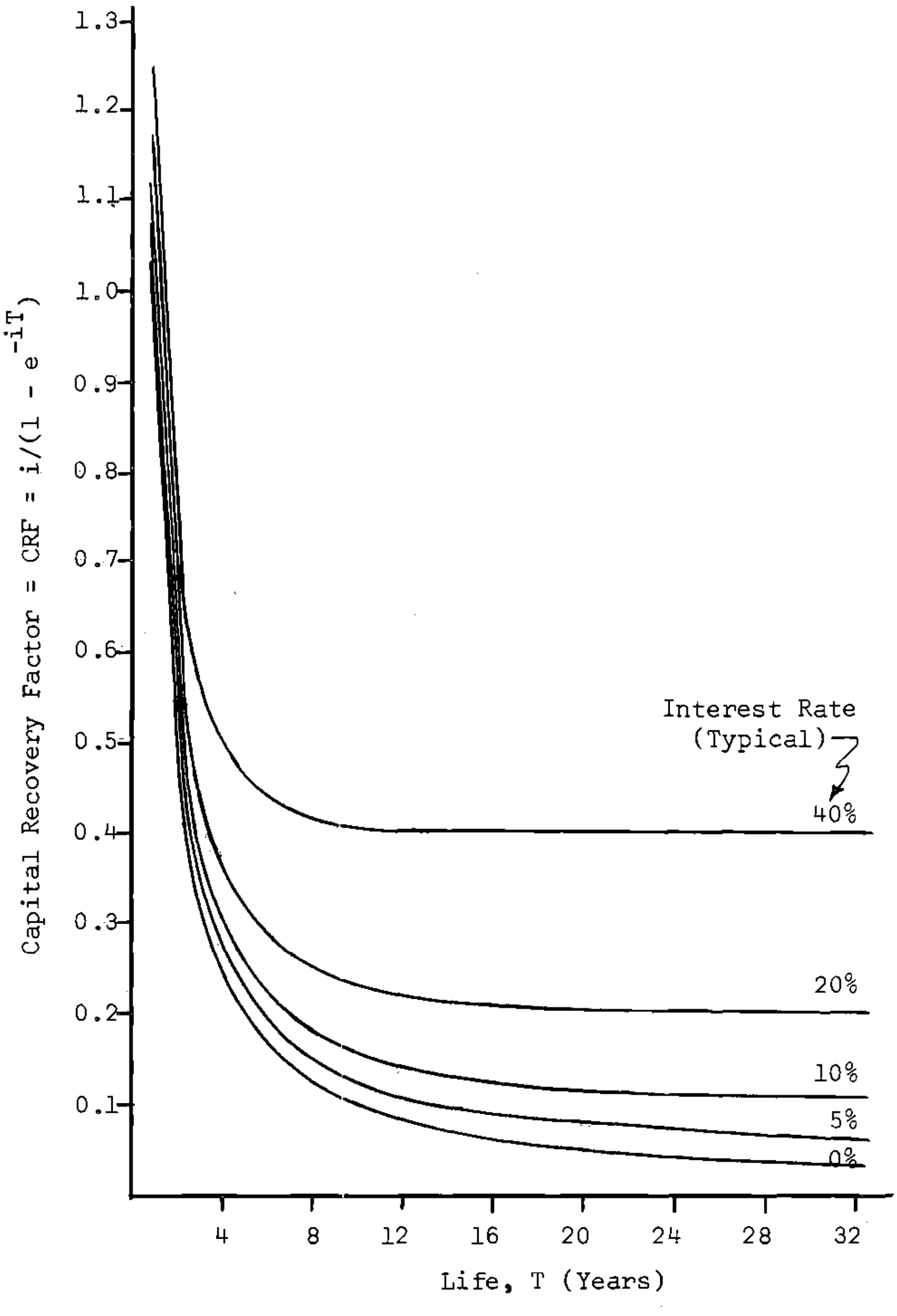


Figure 5. Capital Recovery Factor Versus Life of Project

interest which is examined for fewer conditions is R_{crm} = E(CRF)/CRF at Modal Life.

Most of the calculations for determining R_{cr} and R_{crm} were done using the Gaussian mechanical quadrature method for integration. Appendix G contains a sample computer program for calculating R_{cr} and R_{crm}. The particular program shown is for the triangular life distribution. This Gaussian method was also used for most other calculations involving voluminous integration which were undertaken in this study.

Figures 6-17 show graphically R_{cr} and R_{crm} for distribution types and parameters and interest rates selected to cover a wide range of conditions. R_{crm}, which differs from R_{cr} only for non-symmetrical distributions, is sometimes shown only for 10 per cent interest for clarity and because the results for 5 per cent interest and 20 per cent interest closely correspond to the 10 per cent results.

Figures 6-17 can serve as reference to the practitioner who wants to determine the relative effect on the CRF of quantitatively considering the dispersion of life through calculation of expectations versus not considering that dispersion and rather basing calculations on expected life or modal life. What is actually a notable or significant difference between E(CRF) and CRF at Expected Life or CRF at Modal Life as shown by the value of R_{cr} and R_{crm} is a somewhat subjective question which depends upon many conditions inherent in the particular projects subject to analysis. Some of the most important of these conditions are the degree of confidence in the estimates used in the analysis, the degree of similarity of the distributions of lives of the various alternatives being considered, the effect of the CRF in the analysis result for each

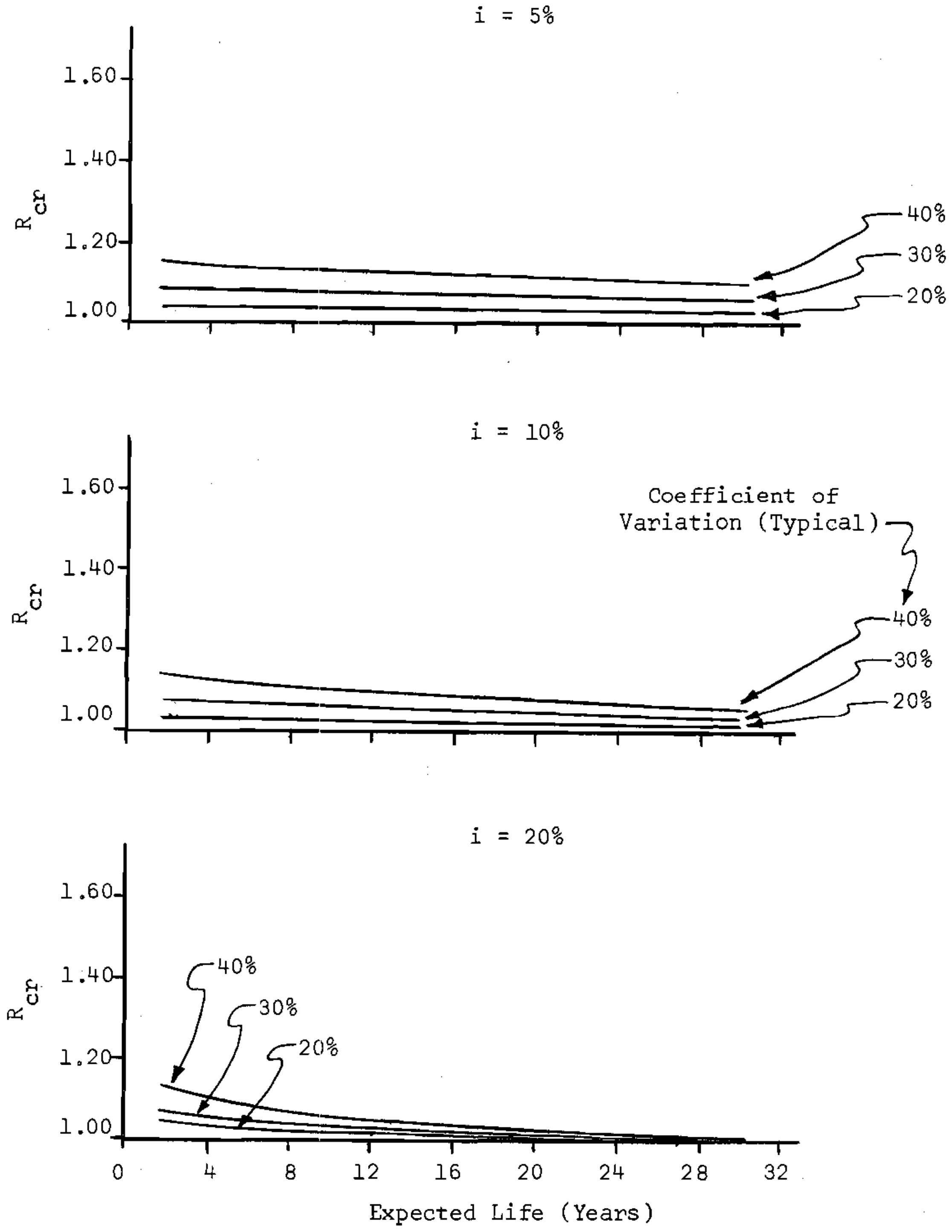


Figure 6. Effect of Normally Distributed Life on R_{cr}

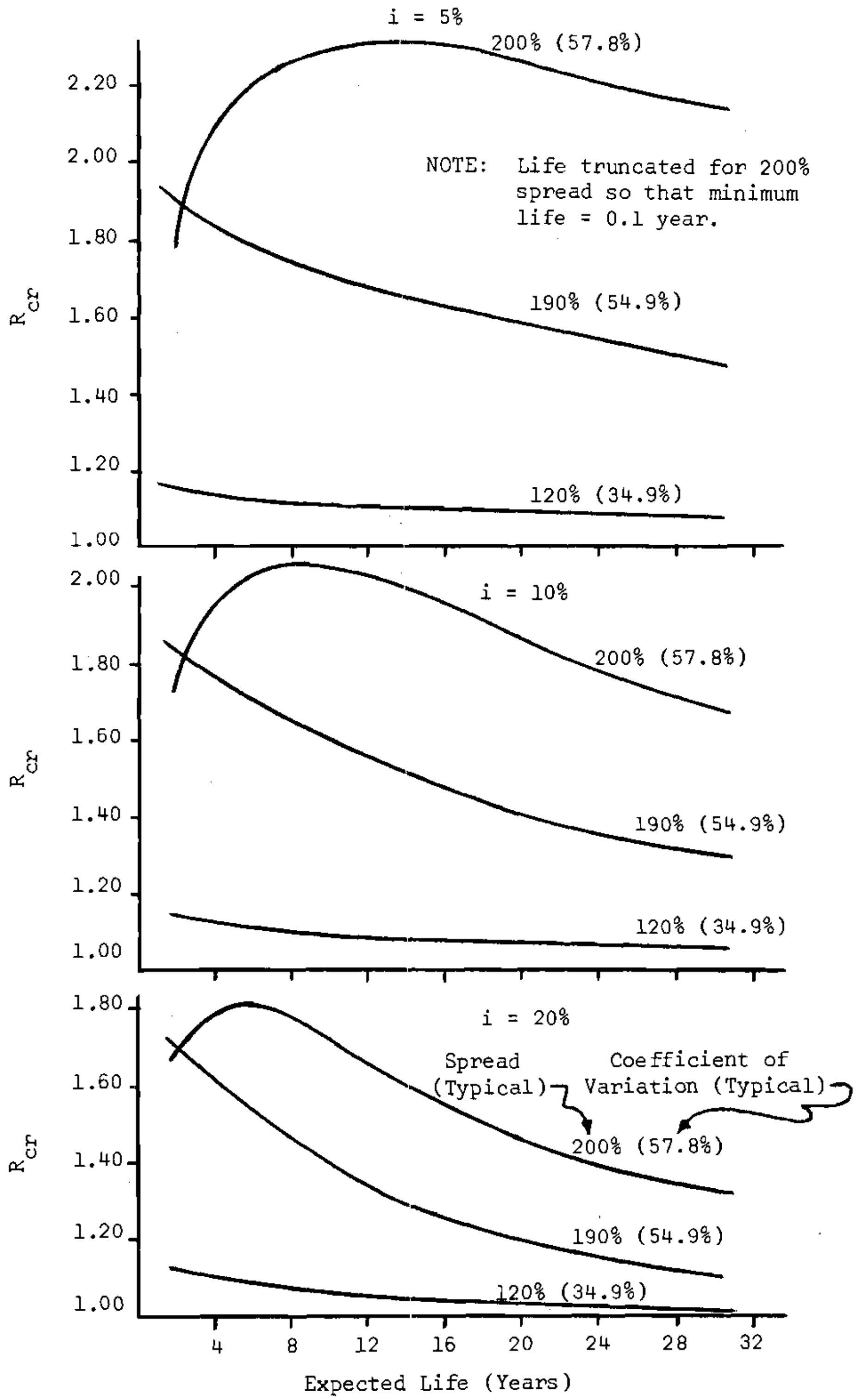


Figure 7. Effect of Uniformly Distributed Life on R_{cr}.

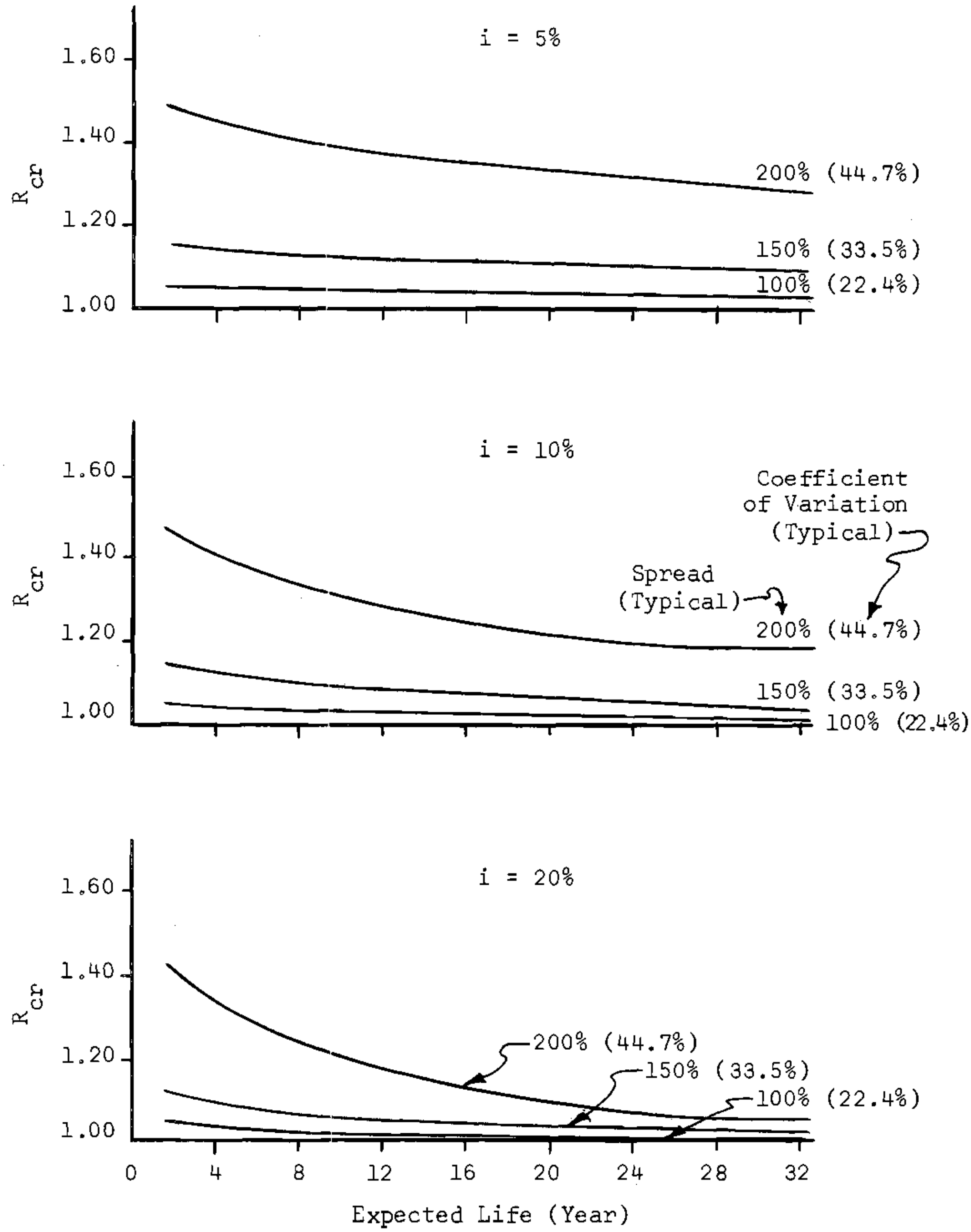


Figure 8. Effect of Beta Distributed Life on R_{cr} for Shape Parameters $\alpha = 1$, $\beta = 1$ (Symmetrical)

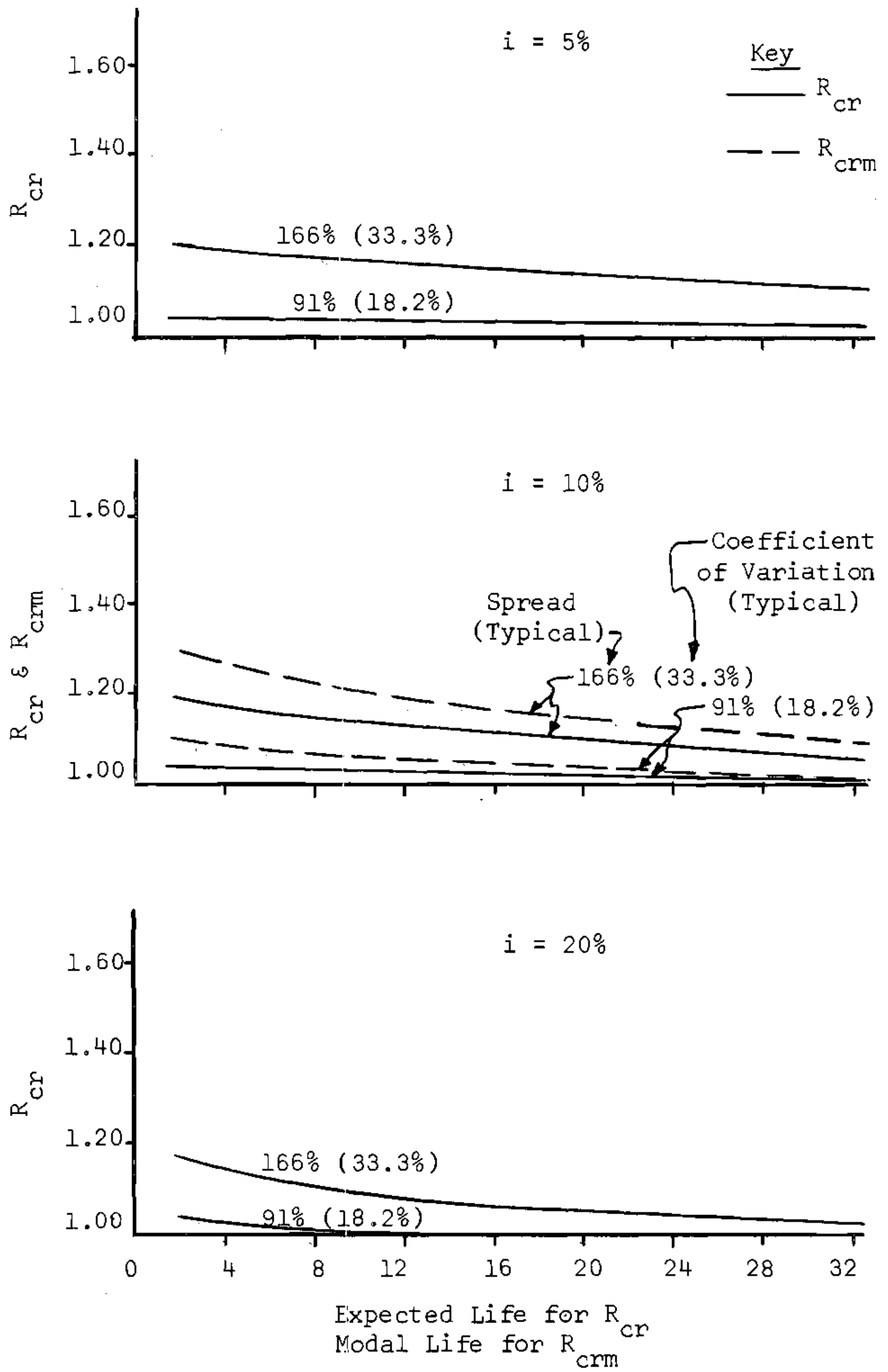


Figure 9. Effect of Beta Distributed Life on R_{cr} and R_{crm} for Shape Parameters $\alpha = 2$, $\beta = 1$ (Left Skewed)

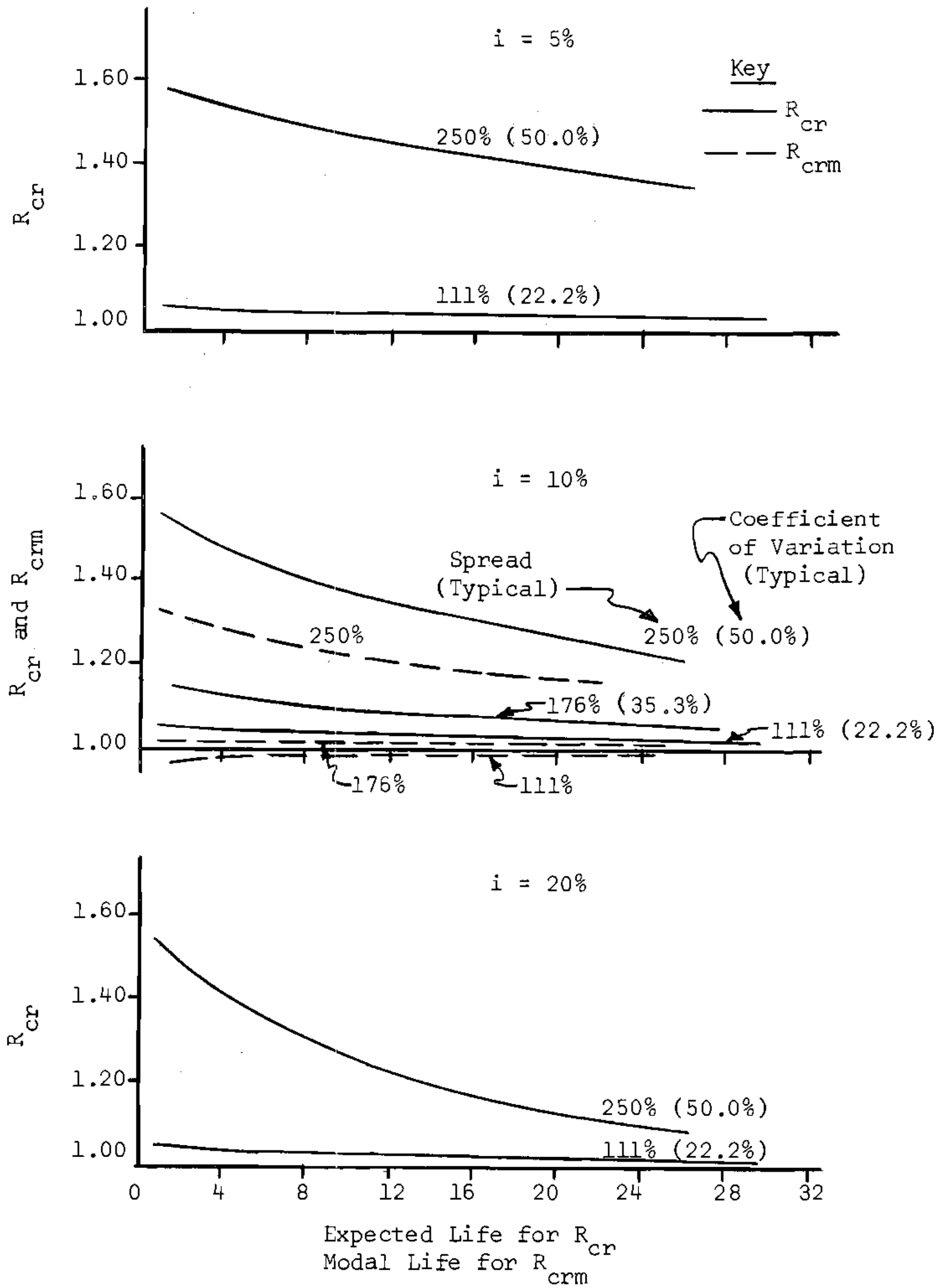


Figure 10. Effect of Beta Distributed Life on R_{cr} and R_{crm} for Shape Parameters $\alpha = 1$, $\beta = 2$ (Right Skewed)

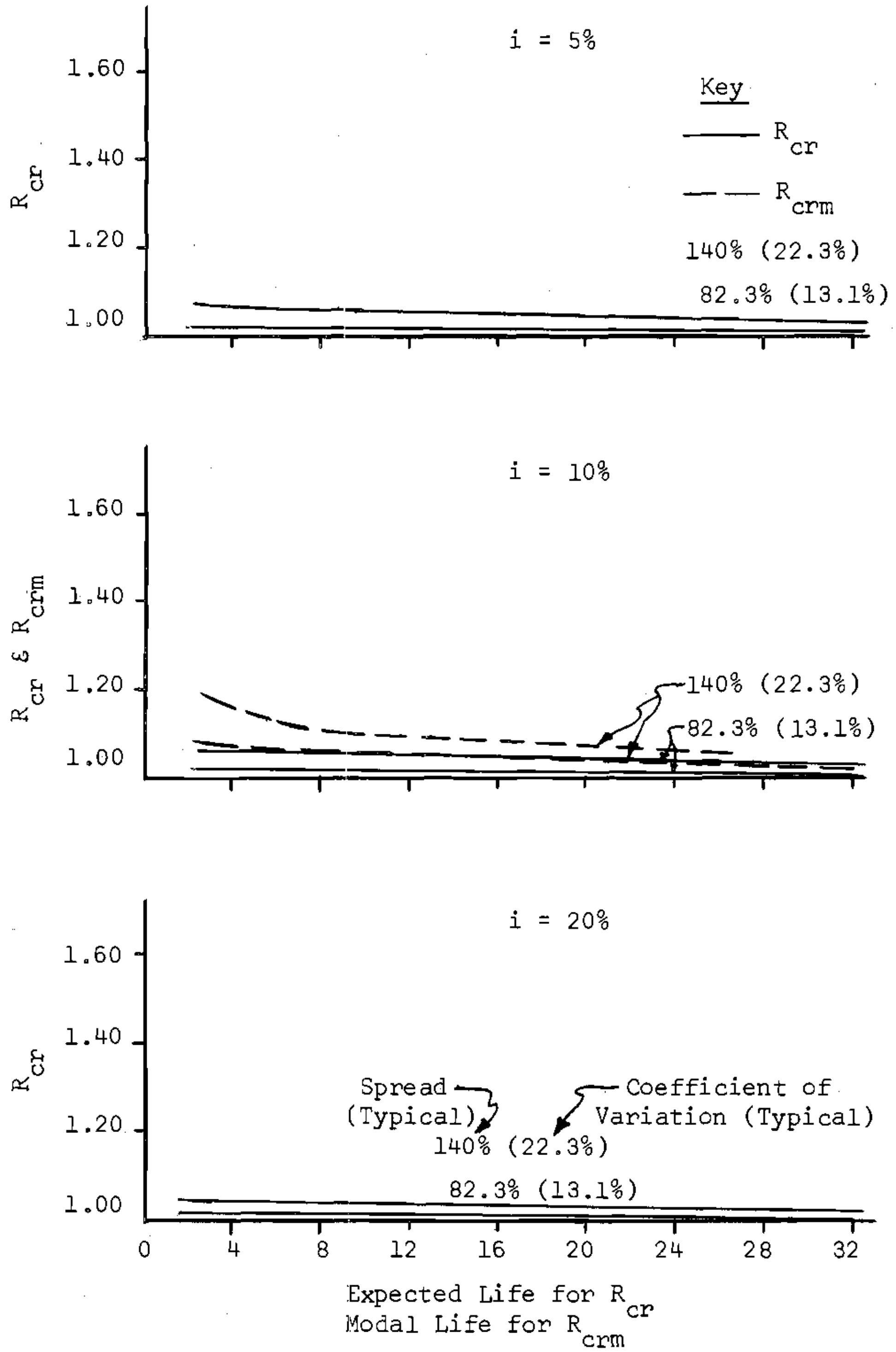


Figure 11. Effect of Beta Distributed Life on R_{cr} and R_{crm} for Shape Parameters $\alpha = 4$, $\beta = 1$ (Left Skewed)

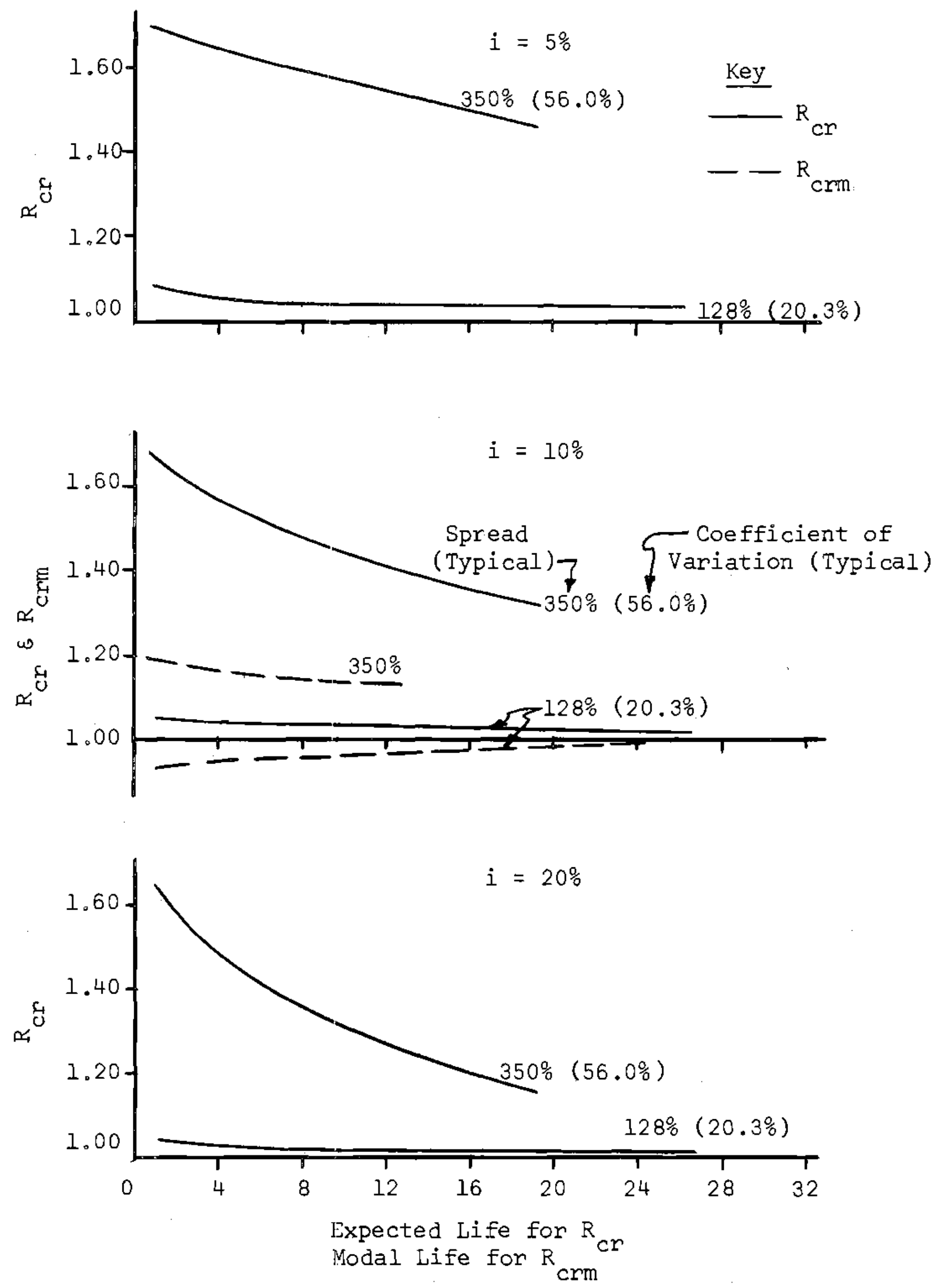


Figure 12. Effect of Beta Distributed Life on R_{cr} and R_{crm} for Shape Parameters $\alpha = 1$, $\beta = 4$ (Right Skewed)

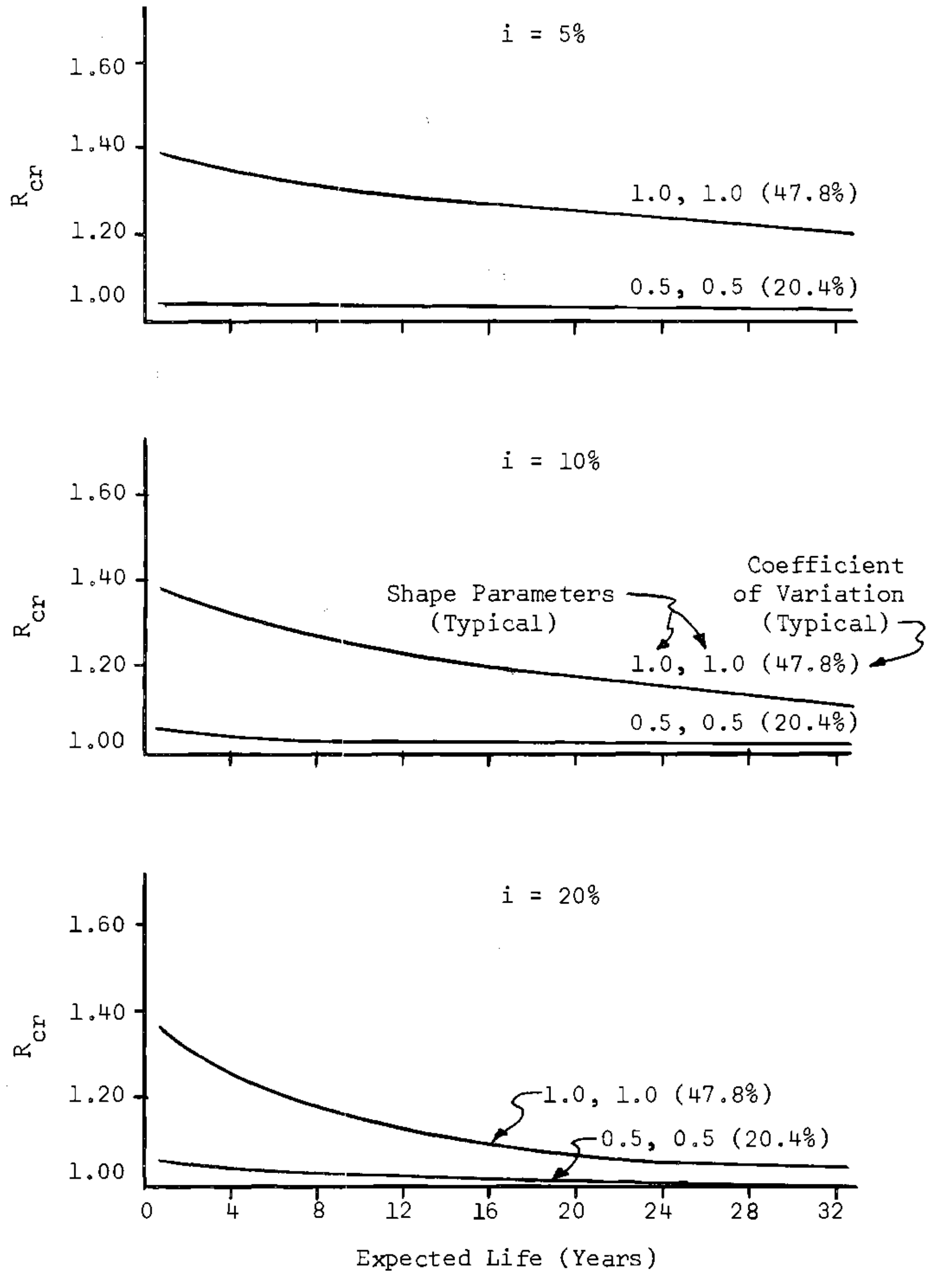


Figure 13. Effect of Triangular Distributed Life on R_{cr} for Shape Parameters G = 1.0, H = 1.0 and G = 0.5, H = 0.5 (Symmetrical)

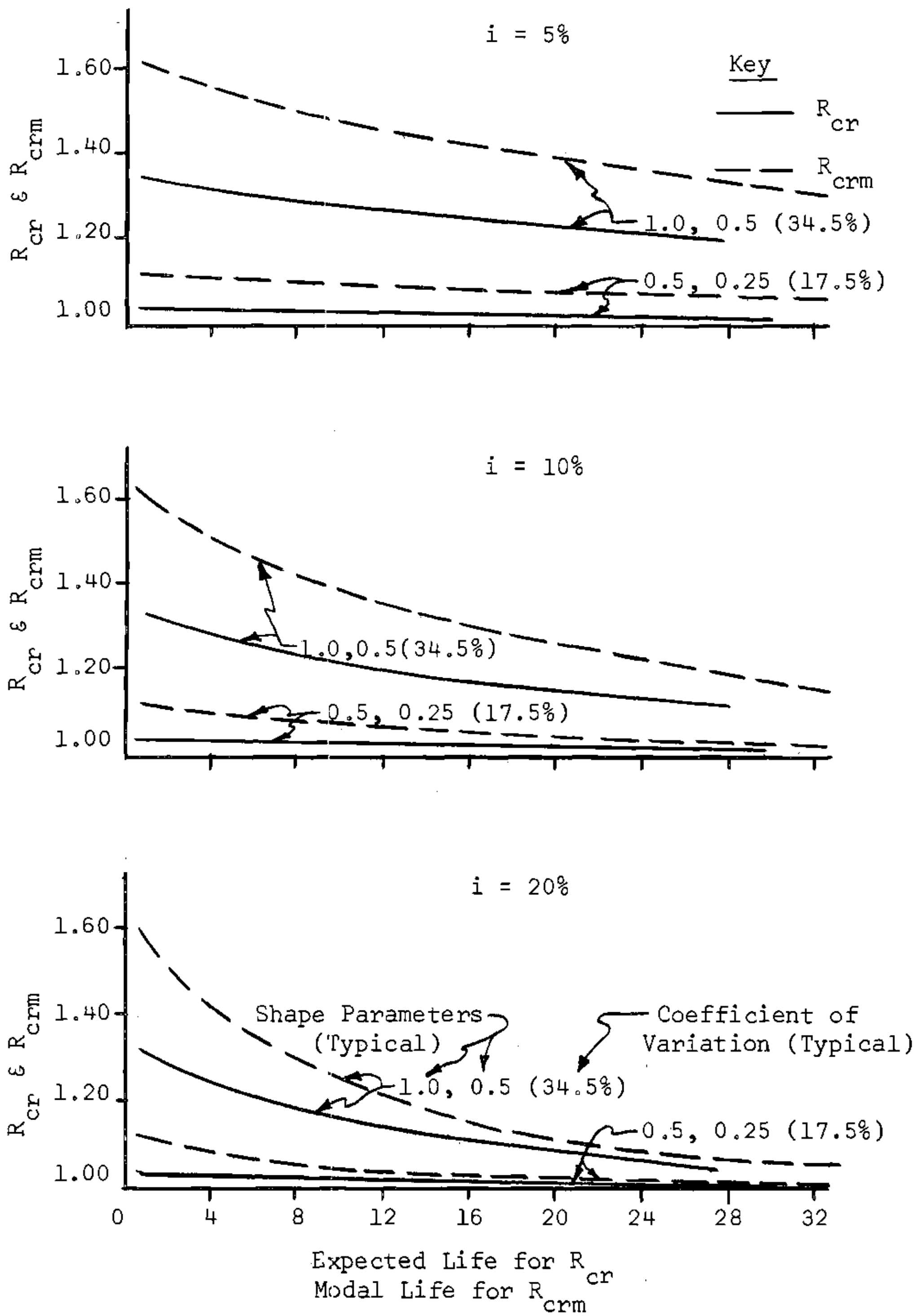


Figure 14. Effect of Triangular Distributed Life on R_{cr} and R_{crm} for Shape Parameters G = 1.0, H = 0.5 and G = 0.5, H = 0.25 (Left Skewed)

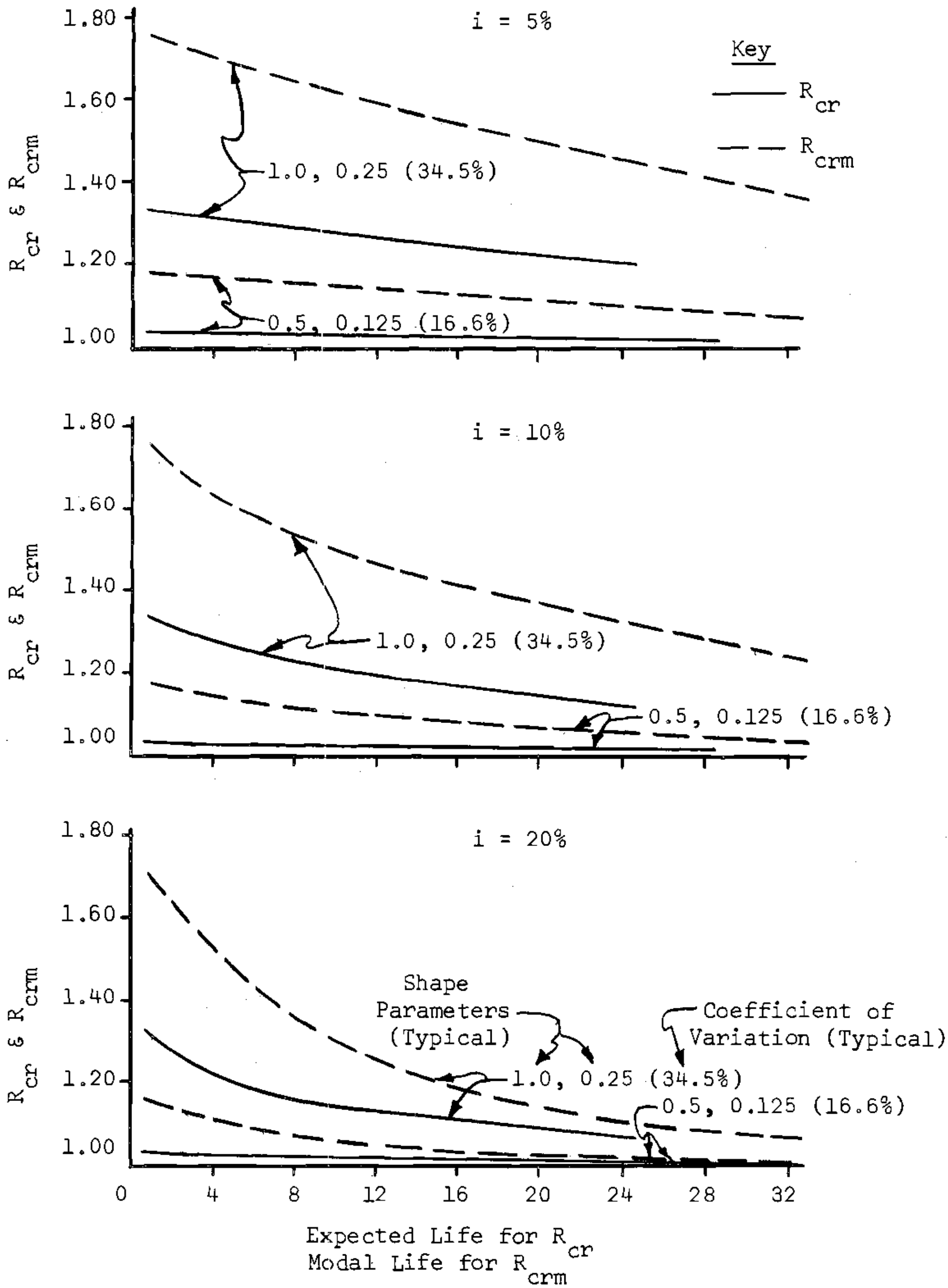


Figure 15. Effect of Triangular Distributed Life on R_{cr} and R_{crm} for Shape Parameters G = 1.0, H = 0.25 and G = 0.5, H = 0.125 (Left Skewed)

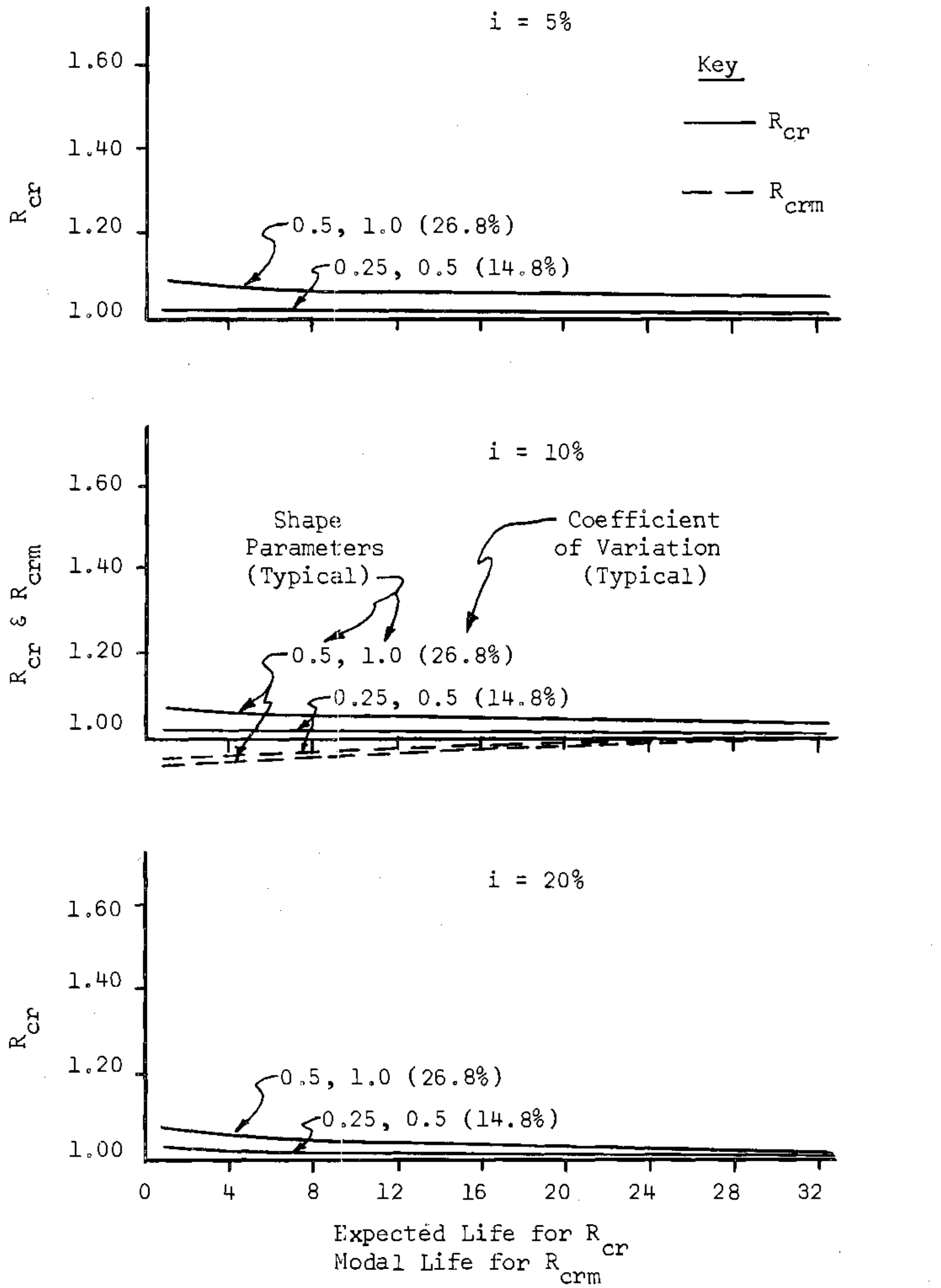


Figure 16. Effect of Triangular Distributed Life on R_{cr} and R_{crm} for Shape Parameters G = 0.5, H = 1.0 and G = 0.25, H = 0.5 (Right Skewed)

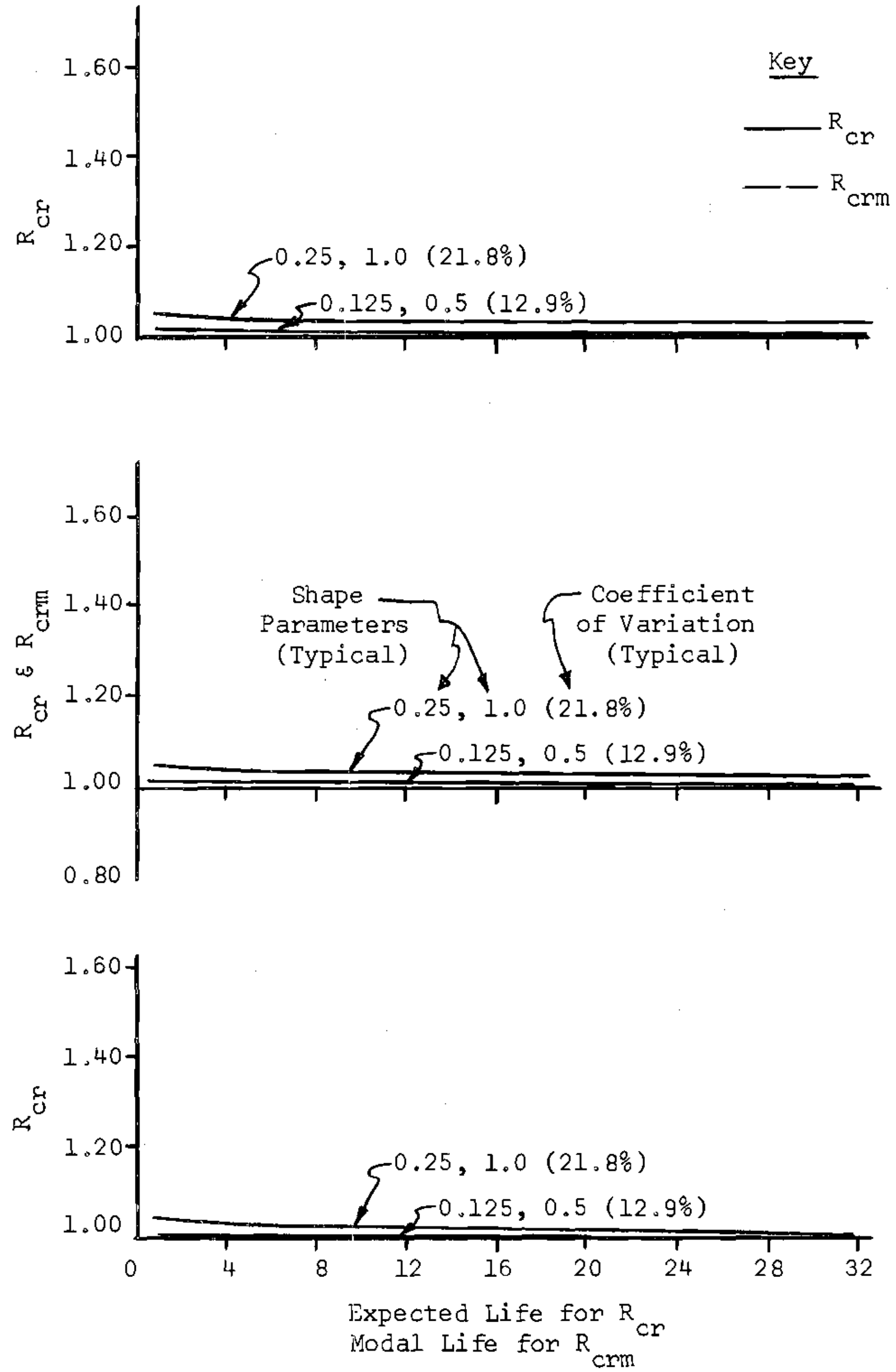


Figure 17. Effect of Triangular Distributed Life on R_{cr} and R_{crm} for Shape Parameters G = 0.25, H = 0.5 and G = 0.125, H = 0.5 (Right Skewed)

alternative, and the economic importance of the alternatives being considered. This matter will be discussed further in the last section of this chapter and in Chapter V.

Because of the wide range of distribution types, distribution variances, and interest rates studied, it will be practical here to make only broad generalizations on amounts and trends of effects on R_{cr} and R_{crm} for each of the distributions. Effects on R_{cr} will be examined first and followed by effects on R_{crm}.

R_{cr} = E(CRF)/CRF at Expected Life

Overall Observations. Examination of R_{cr} for the various distributions and conditions considered reveals that R_{cr} is always greater than 1.00. That is, quantitative consideration of the dispersion of lives results in an E(CRF) which is always equal to or greater than the corresponding CRF at Expected Life. For a given interest rate, if the life distribution changes so that there is increased probability of short lives occurring, E(CRF) and R_{cr} are increased correspondingly. On the other hand, if the life distribution changes so that there is increased probability of long lives occurring, R_{cr} tends to decrease toward 1.00 as a limit.

For a given life distribution, R_{cr} is highest for short expected lives and then decreases for increasing expected lives. At very low expected lives (about two years), R_{cr} is not appreciably affected by differences in interest rates. However, for longer expected lives, the higher the interest rate the lower the value of R_{cr}.

Normal Distribution (Figure 6). The effect of a normally distributed life on R_{cr} is not too pronounced. Coefficients of variation of

40 per cent result in maximum R_{cr} values of approximately 1.15 at two years expected life. This maximum R_{cr} decreases to approximately 1.04 for a coefficient of variation of 20 per cent.

Uniform Distribution (Figure 7). The effect of a uniformly distributed life on R_{cr}, as shown in Figure 7, is greater than any other of the distributions studied whenever the minimum possible life approaches zero. This is because of the fact that the other distributions entail a very, very low probability of occurrence of a life which is close to zero, whereas the uniform distribution entails an equal probability of all possible lives in the range considered. For purposes of discussion, the measure of dispersion "maximum life minus minimum life/expected life" will be referred to as "spread." A spread of 190 per cent of the expected life (coefficient of variation of 54.9 per cent) can result in a maximum R_{cr} of around 1.90 at two years expected life. A spread of 200 per cent can result in much higher R_{cr} values for asymptotically low expected lives. So that the effect of 200 per cent spread can be shown graphically, the life distribution is adjusted so that the minimum life is no less than 0.1 year. This adjustment is the reason for the R_{cr} curve for 200 per cent spread decreasing for low expected lives. Spreads of 120 per cent result in a maximum R_{cr} of only around 1.15.

Beta Distribution (Figures 8-12). The effect of a Beta distributed life on R_{cr} can be great for symmetrical distributions and for right skewed distributions with wide spreads. Table 1 below contains a summarization of the approximate R_{cr} values for i = 10 per cent at expected lives of 2 years and 15 years. Note that R_{cr} is lowest for left skewed distributions and small coefficients of variation.

Table 1. Typical R_{cr} Values for Beta Distributed Lives and i = 10 Per Cent

Shape Description	Parameters		Spread Per Cent	Coefficient of Variation (Per Cent)	R_{cr} at Expected Lives: 2 Years	15 Years
Symmetrical	1	1	200	44.7	1.45	1.25
			150	33.5	1.14	1.08
			100	22.4	1.05	1.03
Left Skewed	2	1	166	33.3	1.18	1.10
			91	18.2	1.03	1.02
Left Skewed	4	1	140	22.3	1.07	1.04
			82	13.1	1.02	1.01
Right Skewed	1	2	250	50.0	1.53	1.30
			176	35.3	1.13	1.07
			111	22.2	1.05	1.03
Right Skewed	1	4	350	56.0	1.64	1.37
			220	35.2	1.13	1.06
			128	20.3	1.04	1.02

Triangular Distribution (Figures 13-17). The effect of the triangular distribution closely corresponds to the Beta distribution for similar amounts of skewness. Table 2 below is a summarization of the approximate R_{cr} values as determined for i = 10 per cent at expected lives of 2 years and 15 years.

R_{crm} = E(CRF)/CRF at Modal Life

Rather than elaborate on quantitative results, only broad observations on how R_{crm} compares with R_{cr} will be made.

Beta Distribution (Figures 9-12). For the left skewed cases in Figures 9 and 11, R_{crm} deviates roughly two times as much above 1.00 as does R_{cr}. For the right skewed cases in Figures 10 and 12, there is no

Table 2. Typical R_{cr} Values for Triangular Distributed Lives and i = 10 Per Cent

Shape Description	Shape Parameters G	H	Coefficient of Variation (Per Cent)	R_{cr} at Expected Lives 2 Years	15 Years
Symmetrical	1.0	1.0	40.8	1.35	1.20
	0.5	0.5	20.4	1.04	1.03
Left Skewed	1.0	0.5	34.5	1.34	1.18
	0.5	0.25	17.5	1.03	1.02
Left Skewed	1.0	0.25	36.3	1.30	1.17
	0.5	0.125	16.6	1.03	1.01
Right Skewed	0.5	1.0	26.8	1.07	1.04
	0.25	0.50	14.8	1.02	1.01
Right Skewed	0.25	1.0	21.8	1.04	1.02
	0.125	0.5	12.9	1.02	1.01

pattern except that R_{crm} is greater than 1.0 for cases of greatest spread and becomes slightly less than 1.0 for cases of intermediate spread.

Triangular Distribution (Figures 14-17). For the left skewed cases in Figures 14 and 15, R_{crm} deviates roughly two times as much above 1.00 as does R_{cr}. For the right skewed cases in Figures 16 and 17, R_{crm} is about as much below 1.00 as R_{cr} is above 1.00. R_{crm} tends to approach 1.00 for high modal lives.

Capital Recovery Cost and Capital Recovery Cost Factor

Capital recovery cost is defined for use here as the equivalent uniform annual cost of depreciation and interest on investment. To show the relation of the capital recovery cost to the CRF, capital recovery cost for assumed certainty is computed as:

$$CRC(\$) = (P-S)(i/1-e^{-iT}) + S(i) \tag{14}$$

where CRC (\$) = capital recovery cost,

P = first cost or investment,

S = salvage value,

i = nominal rate of interest,

T = life (years), and

$i/1-e^{-iT}$ = capital recovery factor (CRF).

To keep generality, capital recovery cost will be referred to as proportion of the first cost, P, thus creating what will be called a capital recovery cost factor. This can be obtained by dividing Equation (14) by P:

$$CRC(\$)/P = (1-S/P)(i/1-e^{-iT}) + (S/P)(i), \tag{15}$$

where CRC(\$)/P is the capital recovery cost factor which will be denoted hereafter as CRC. The effect of the salvage value on CRC is described below.

Constant Salvage Value

If the salvage value is constant throughout the life of a project, CRC is a weighted average of the capital recovery factor, $i/1-e^{-iT}$, and the interest rate, i, with the weights depending on S/P. Figure 5, which shows the behavior of the CRF for various project lives and interest rates, at the same time depicts the CRC for the case of constant zero salvage values. Figures 18 and 19 show the effect of various constant

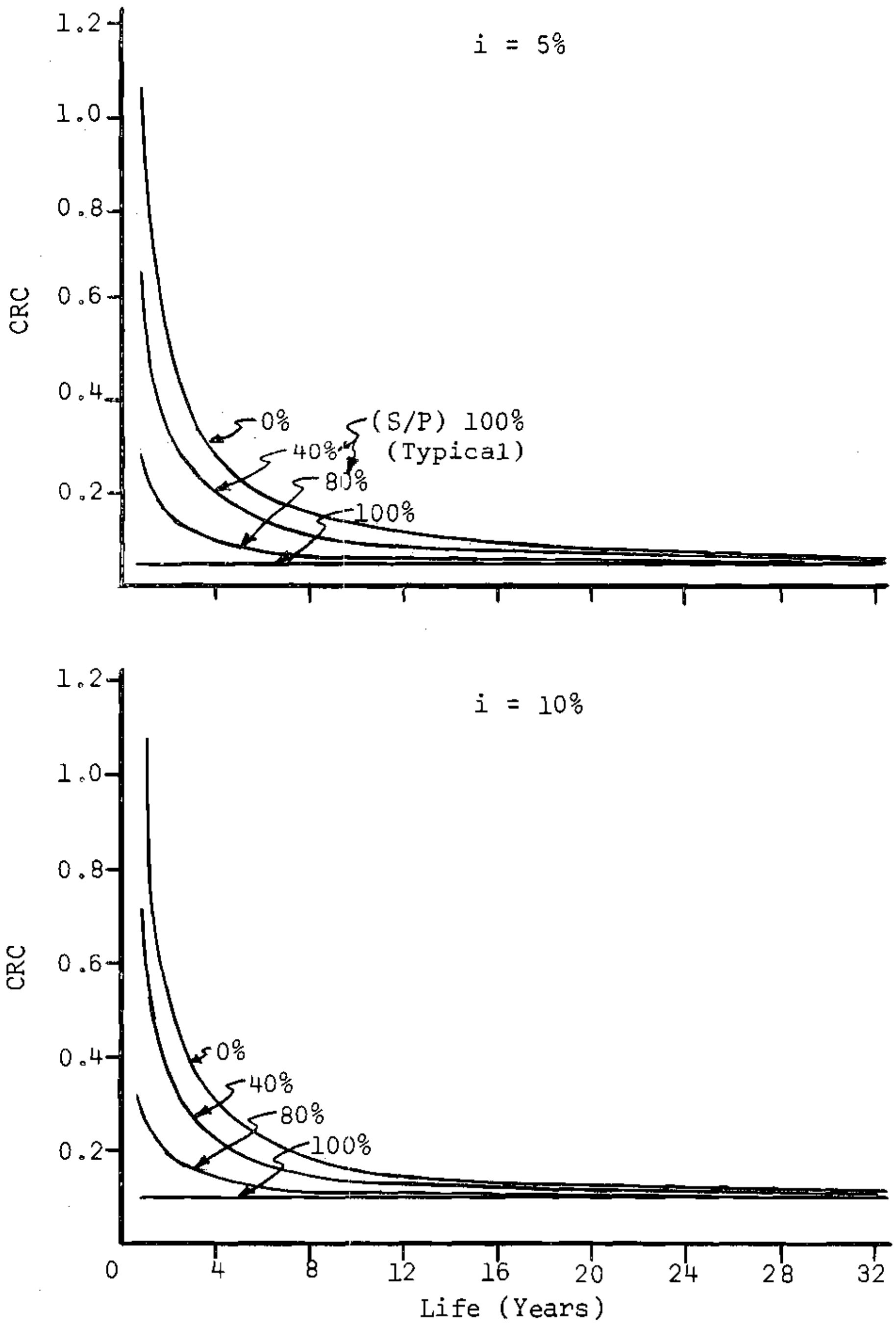


Figure 18. Effect of Constant Salvage Value Throughout Life on Capital Recovery Cost Factor

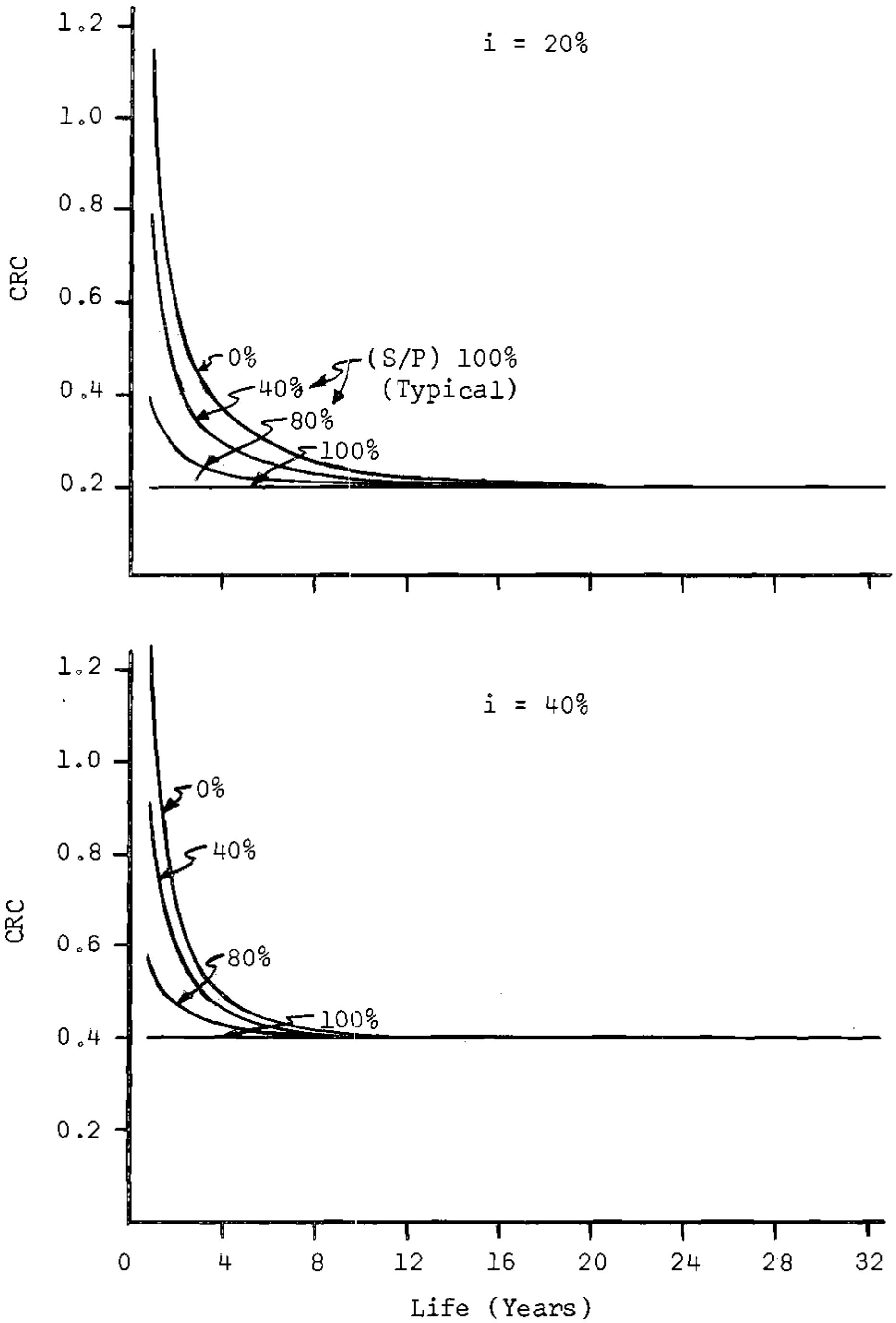


Figure 19. Effect of Constant Salvage Value Throughout Life on Capital Recovery Cost Factor

S/P conditions (expressed as a per cent) on CRC. Note that the CRC is a minimum (and also a constant equal to i) for S/P = 100 per cent. Note also that the CRC is a maximum (equal to CRF) for S/P = 0 per cent.

Salvage Value Decreasing with Time

If the salvage value is a function of time, then the CRC depends not only upon the CRF (and hence the interest rate and life) but it also depends upon the nature of the salvage function itself. Thus, if S_t is the salvage value at time t, then the CRC which is a function of S_t, called CRC_t, may be expressed as:

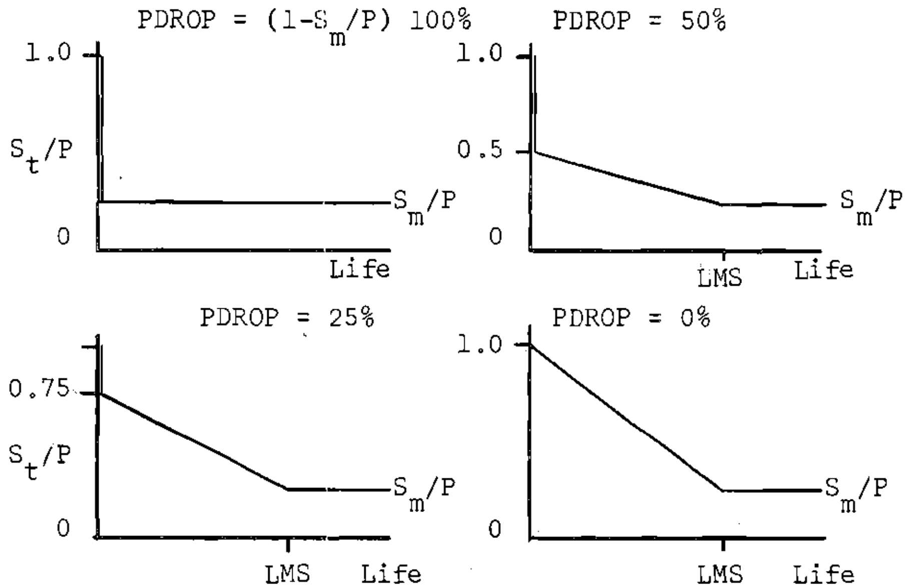
$$CRC_t = (1-S_t/P)(i/1-e^{-iT}) + (S_t/P)(i). \qquad (16)$$

In Appendix B, the results of an investigation of the effect on CRC_t of a range of two classes of salvage value functions, the straight line and exponential, are shown. Both classes involve non-increasing salvage value functions only.

Figure 20 graphically depicts the various conditions that are studied for both straight line salvage value functions and exponential salvage value functions. Shown on the figure are the general equations for these salvage value functions. Note that initial lump drops in salvage values (called PDROP and expressed as percentage of P) were taken into consideration. The bottom of Figure 20 further describes the conditions studied. The only symbol which has not been introduced previously is LMS, which represents the life at the point where the minimum salvage value is reached. In the case of exponential salvage value functions,

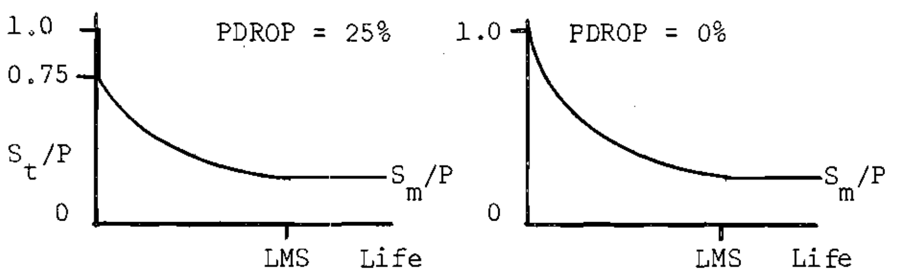
STRAIGHT LINE SALVAGE FUNCTIONS

$$S_t/P = 1\text{-PDROP} - [(1\text{-PDROP-}S_m/P)t]/\text{LMS};\ t \le \text{LMS}$$
$$= S_m/P;\ t > \text{LMS}$$



EXPONENTIAL SALVAGE FUNCTIONS

$S_t/P = S_m/P + (1\text{-PDROP-}S_m/P)e^{-at}$, where a is a constant chosen so that e^{-at} is ≤ 0.01 when life = LMS



Key

P = Investment
PDROP = Initial drop in salvage value as % of P
LMS = Life at point where minimum salvage is reached

Conditions Considered

i = 5%, 10%, 20%
S_m/P = 0, 0.20, 0.40
LMS = 5, 10, 20 years

Figure 20. Effect of Various Salvage Value Functions on CRC_t -- Function Shapes and Other Conditions Considered

the true minimum is actually never reached, so LMS is arbitrarily taken to be the life at which the decline in salvage value becomes 99 per cent of the total decline from P to the minimum salvage value, S_m. Appendix G contains the computer program used for determining the effect of various straight line salvage functions on CRC_t. The program for determining the effect of various exponential salvage value functions is very similar.

Figures 62-64 in Appendix B graphically show the effect of various straight line salvage value functions and interest rates on CRC_t. Figures 65-67, also in Appendix B, show similar information for various exponential salvage value functions and interest rates. Note that the less the PDROP, the lower is the CRC_t for all lives up to LMS. Note also that the higher the minimum salvage value expressed as a proportion of P, S_m/P, the less difference which is caused by various LMS values. These latter observations reflect what seems reasonable in general. The next section combines consideration of the salvage value function and life distribution.

Combined Effect of Salvage Value Function and Life Distribution on Expected Capital Recovery Cost

The final step in the consideration of CRC_t is to discuss the general effect of various distributions of life on its expected value, $E(CRC_t)$. For the completely general case, $E(CRC_t)$ is calculated by the relation:

$$E(CRC_t) = \int_A^Z [(1-S_t/P)(i/1-e^{-iT}) + (S_t/P)(i)]\, f(T)dT, \qquad (17)$$

where all the symbols have previously been defined. It can be seen that the $E(CRC_t)$ is determined by the values of the salvage value function over time as weighted by the probability of all possible times (lives) occurring.

The salvage value as a function of time, S_t/P, may take any form, a few of which were investigated as described in the previous section. Also, there are multitudinous life distributions which can conceivably occur in practice. Hence, the number of possible combinations of S_t/P and $f(T)$ which could be studied to determine the effect on $E(CRC_t)$ is prohibitively large. The study of the effect of different life distributions on $E(CRF)$ relates the extreme effect of dispersion of project life on $E(CRC_t)$. For a given life distribution, $E(CRC_t)$ is as great as $E(CRF)$ only when there is a constant 0 salvage value or a very long life. In Figures 18 and 19, the CRC_t curves for 0 per cent salvage value are also curves for CRF. It can be seen that the CRC_t for all salvage values greater than zero is equal to or less than the CRF for a given life and interest rate. Hence, for a given life distribution and interest rate, $E(CRC_t)$ for all positive salvage values is equal to or less than $E(CRF)$.

Effect of Various Life Distributions on Expected Present Value Factor

The present value factor is a key factor for measuring the worth of periodic operating results of a project or series of projects. It is the factor which takes into account the timing of a series of uniform cash flows (receipts or disbursements) through the discounting of those cash flows. If the cash flow is constant throughout the life of a project, then the present value of that flow is directly proportional to the present value factor. Figure 21 shows the behavior of the present value factor, $(1-e^{-iT})/i$, for a range of lives and interest rates. Note that the factor is steeply sloped for low lives and interest rates and becomes almost constant for sufficiently high lives and interest rates.

This section is a rather detailed examination of the effect of different life distributions on the expected present value factor, E(PV). E(PV) is calculated by the relation:

$$E(PV) = \int_A^Z (1-e^{-iT})/i \cdot f(T)dT \qquad (18)$$

where $(1-e^{-iT})/i$ is the present value factor for the continuous compounding, continuous flow case and all other symbols have previously been defined.

The main intent of this study is to produce information which will help the practitioner determine when it is worthwhile to quantitatively consider dispersion of life through calculating E(PV) rather than calculating PV based on assumed certain expected life or modal life. The main ratio to be used as a basis for comparison is R = E(PV)/PV at Expected Life. Another ratio of interest which is examined for fewer conditions is R_m = E(PV)/PV at Modal Life. Figures 22-34 show graphically R and R_m for distribution types and parameters and interest rates selected to cover the same wide range of conditions as the study of R_{cr} and R_{crm} in the last section.

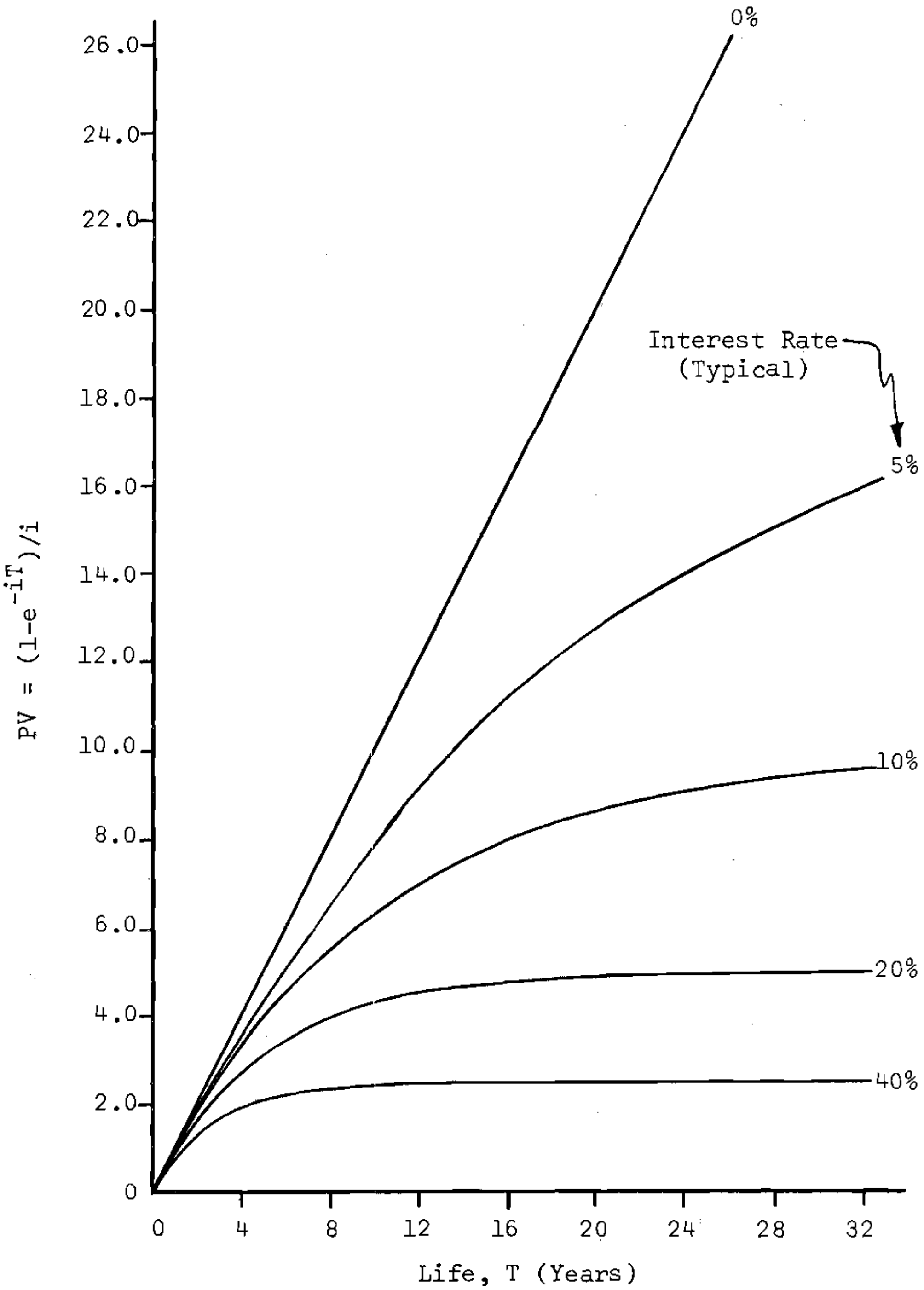


Figure 21. Present Value (Worth) Factor vs. Life

Figures 22-34 can serve as a reference to the practitioner who wants to determine the relative effect on the PV of considering the distribution of life through calculation of expectations versus not considering that distribution and rather basing calculations on expected life or modal life. What is actually a notable difference between E(PV) and PV at Expected Life or PV at Modal Life as shown by how much R or R_m differs from 1.00 is a subjective question which depends on many conditions inherent in the particular situation being analyzed. This will be discussed further in the last main section of this chapter and in Chapter V.

As in the section on R_{cr} and R_{crm}, it will be practical to make only broad generalizations on effects of various life distributions on R and R_m. Effects on R will be examined first and then followed by effects on R_m.

R = E(PV)/PV at Expected Life

Overall Observations. Examination of R for the various distributions and conditions considered reveals that R is always less than 1.00. For a given life distribution, R is close to unity for very short expected lives and then decreases for increasing expected lives until it becomes a minimum at an expected life which depends on the interest rate. The higher the interest rate, the shorter the expected life at which R becomes a minimum. For increasing expected lives beyond the point at which R is a minimum, R increases steadily toward unity.

Normal Distribution (Figure 22). The effect on R of considering a normally distributed life is moderate. A coefficient of variation of 40 per cent results in a minimum R of about 0.94. This minimum R

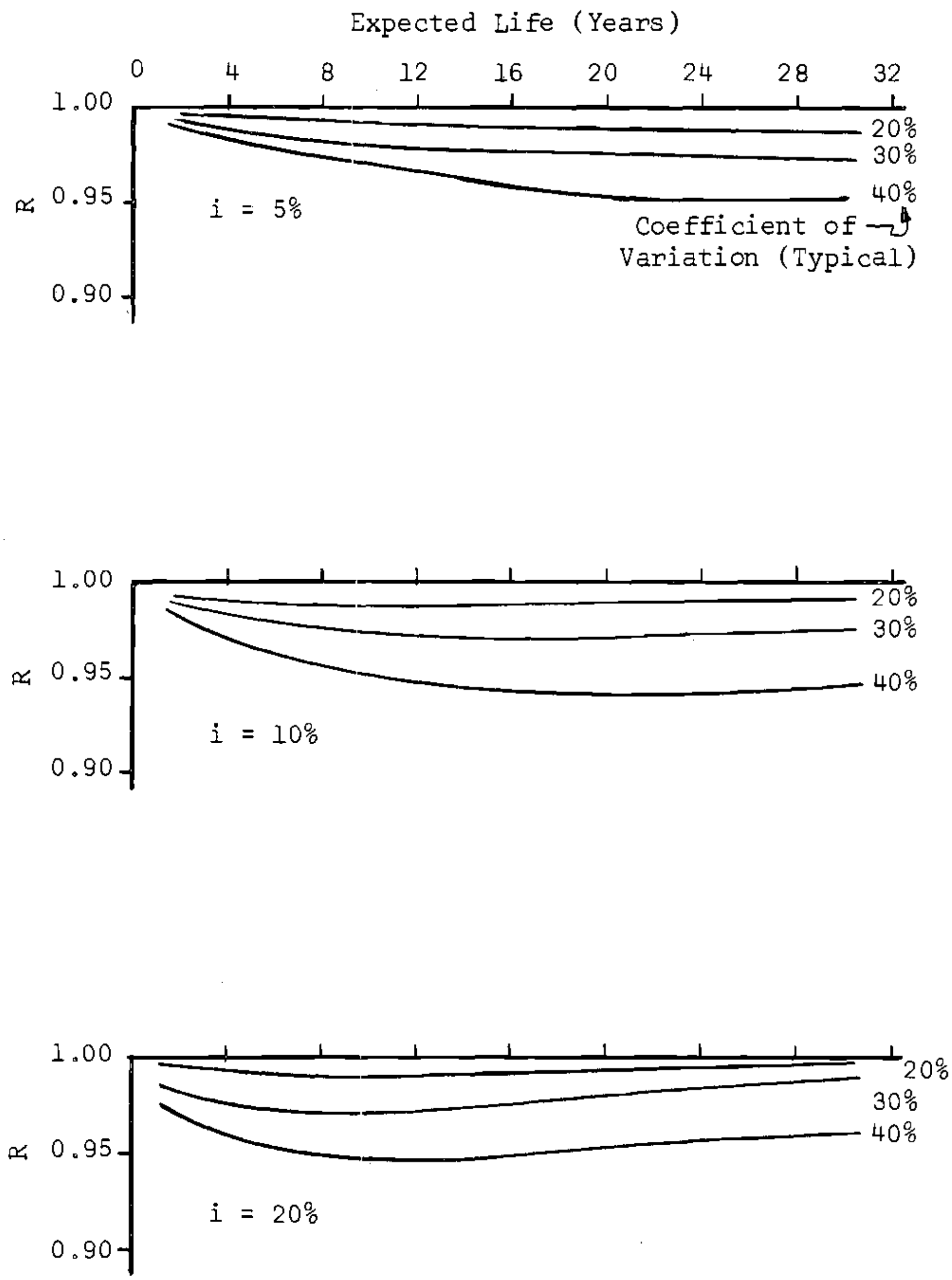


Figure 22. Effect of Normally Distributed Life on R

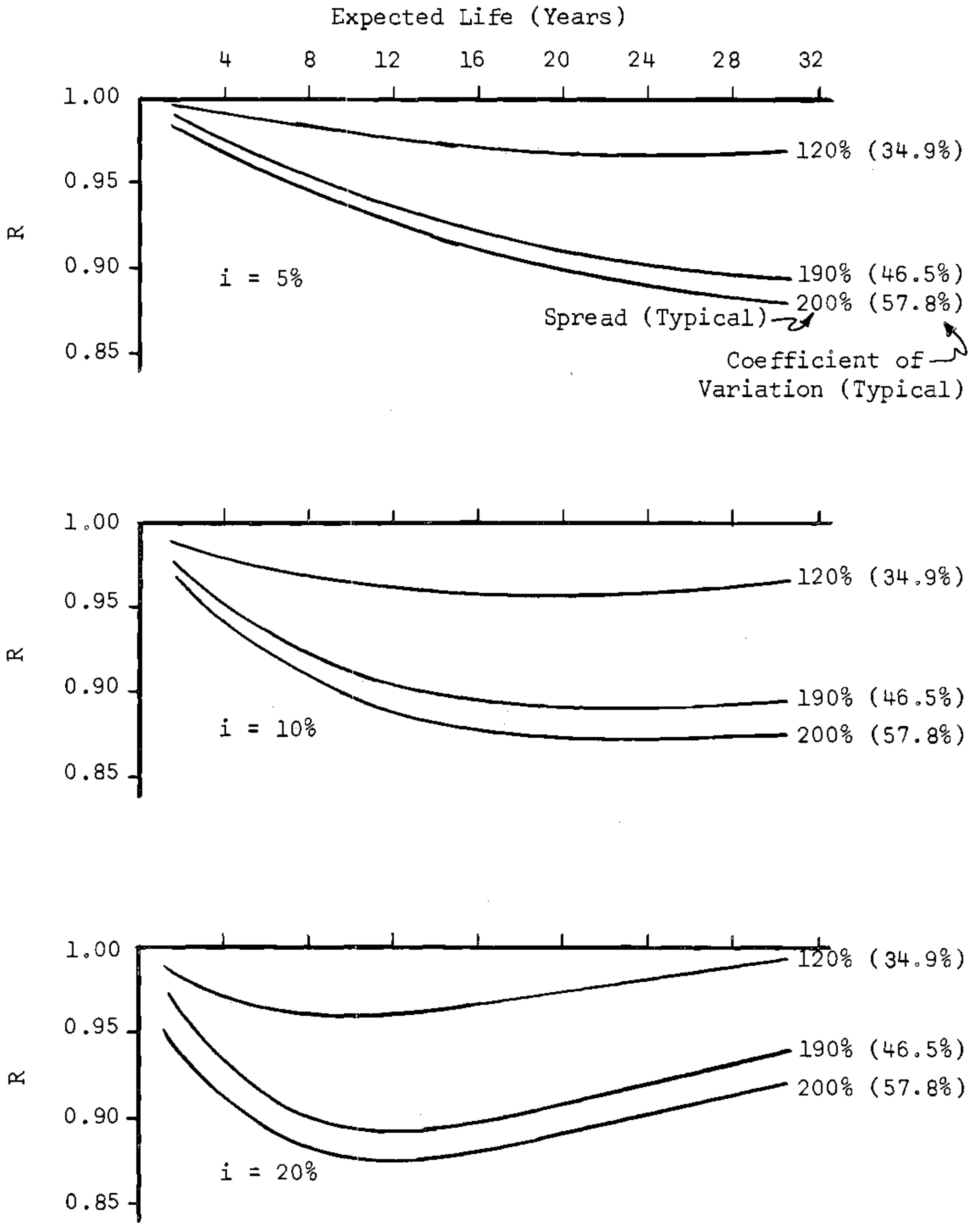


Figure 23. Effect of Uniformly Distributed Life on R

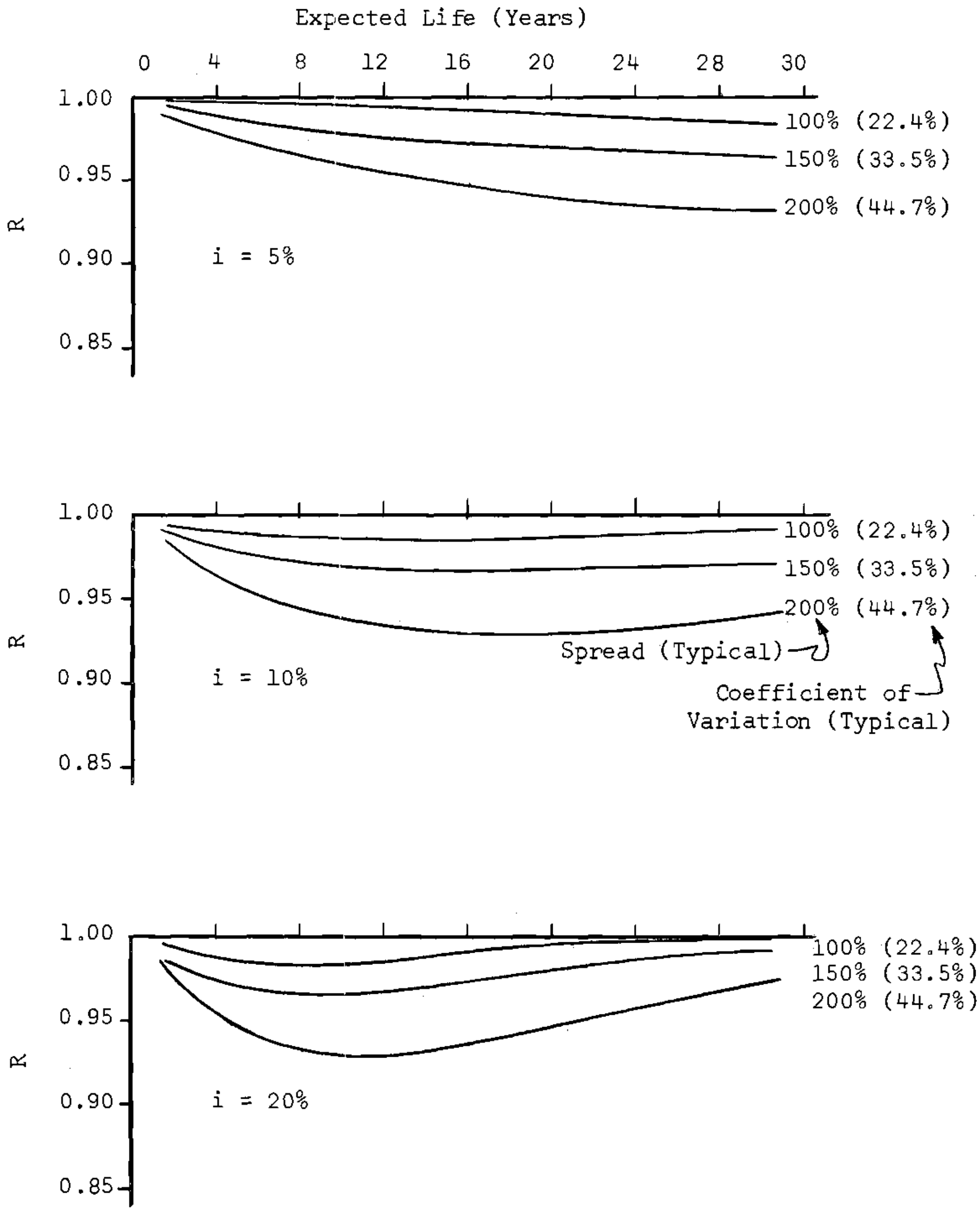


Figure 24. Effect of Beta Distributed Life on R for Shape Parameters $\alpha = 1$, $\beta = 1$ (Symmetrical)

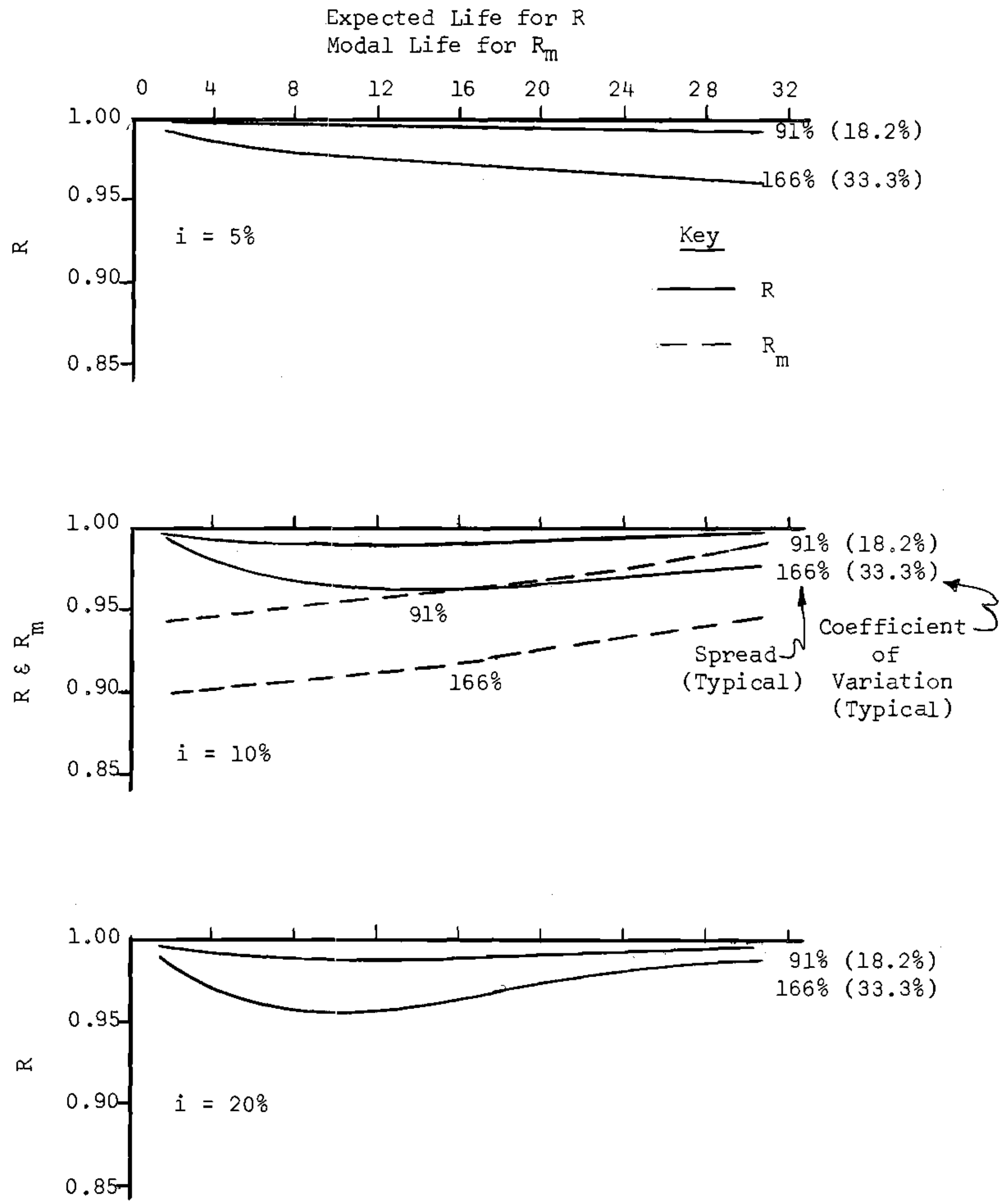


Figure 25. Effect of Beta Distributed Life on R and R_m for Shape Parameters $\alpha = 2$, $\beta = 1$ (Left Skewed)

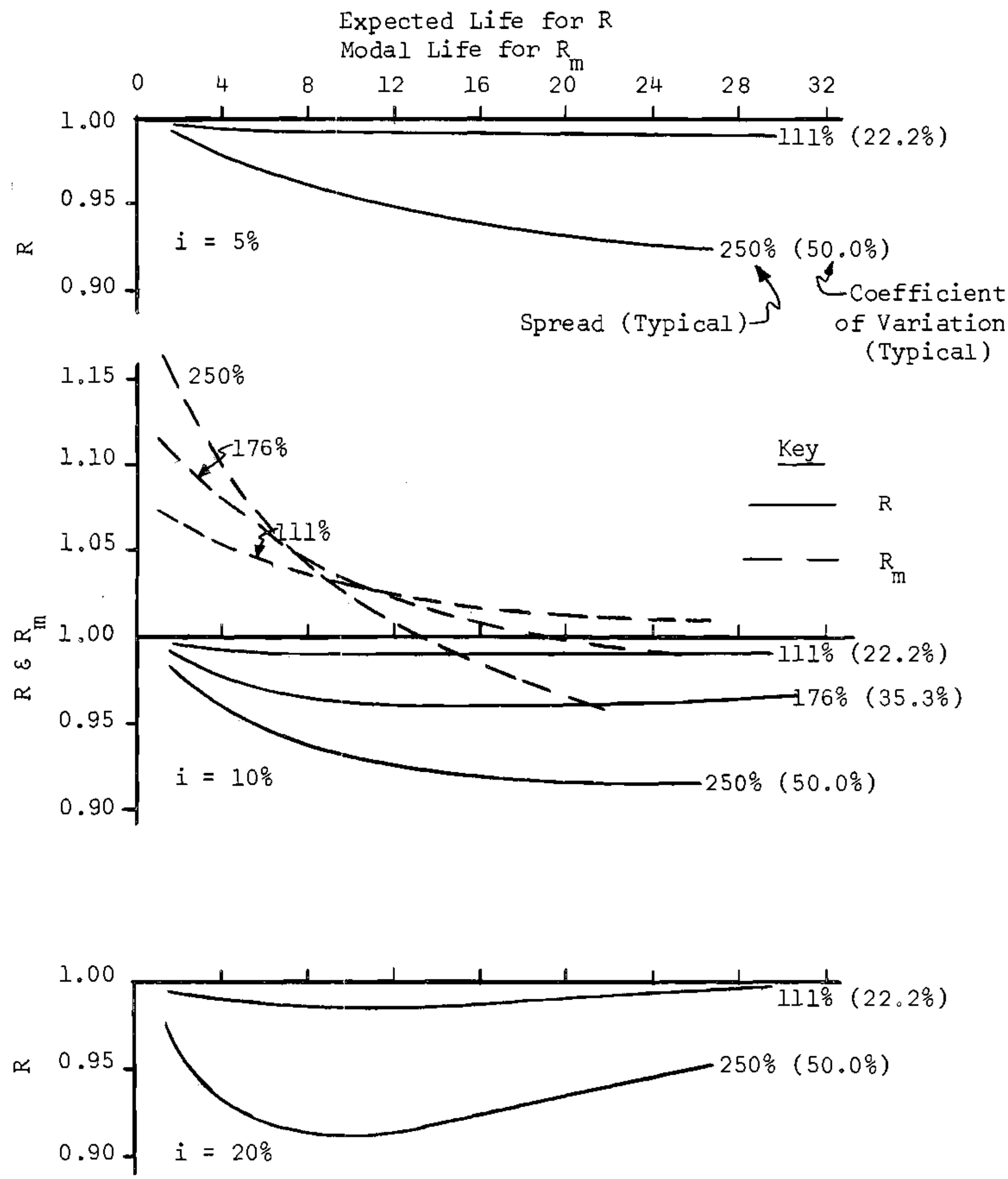


Figure 26. Effect of Beta Distributed Life on R and R_m for Shape Parameters $\alpha = 1$, $\beta = 2$ (Right Skewed)

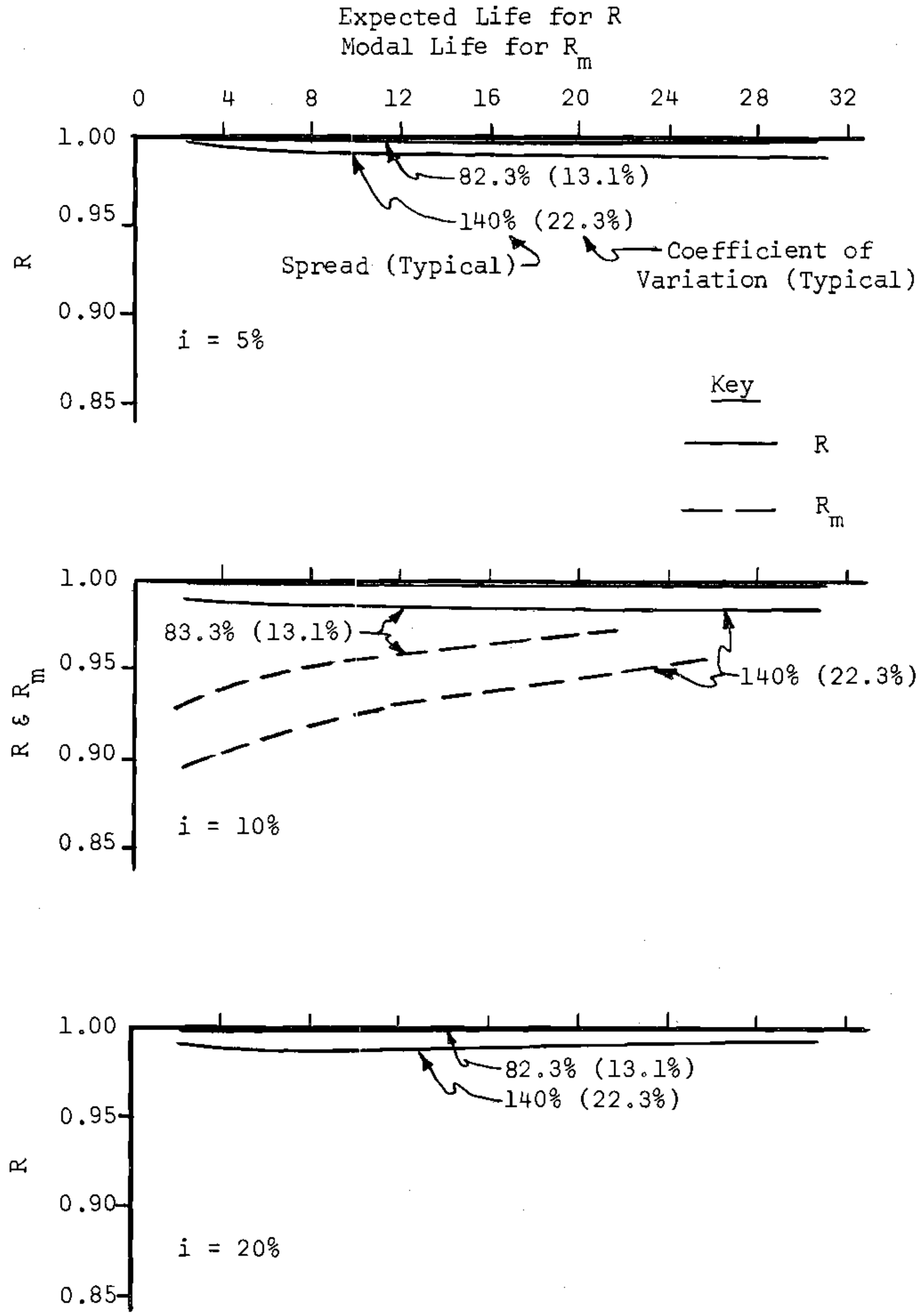


Figure 27. Effect of Beta Distributed Life on R and R_m for Shape Parameters $\alpha = 4$, $\beta = 1$ (Left Skewed).

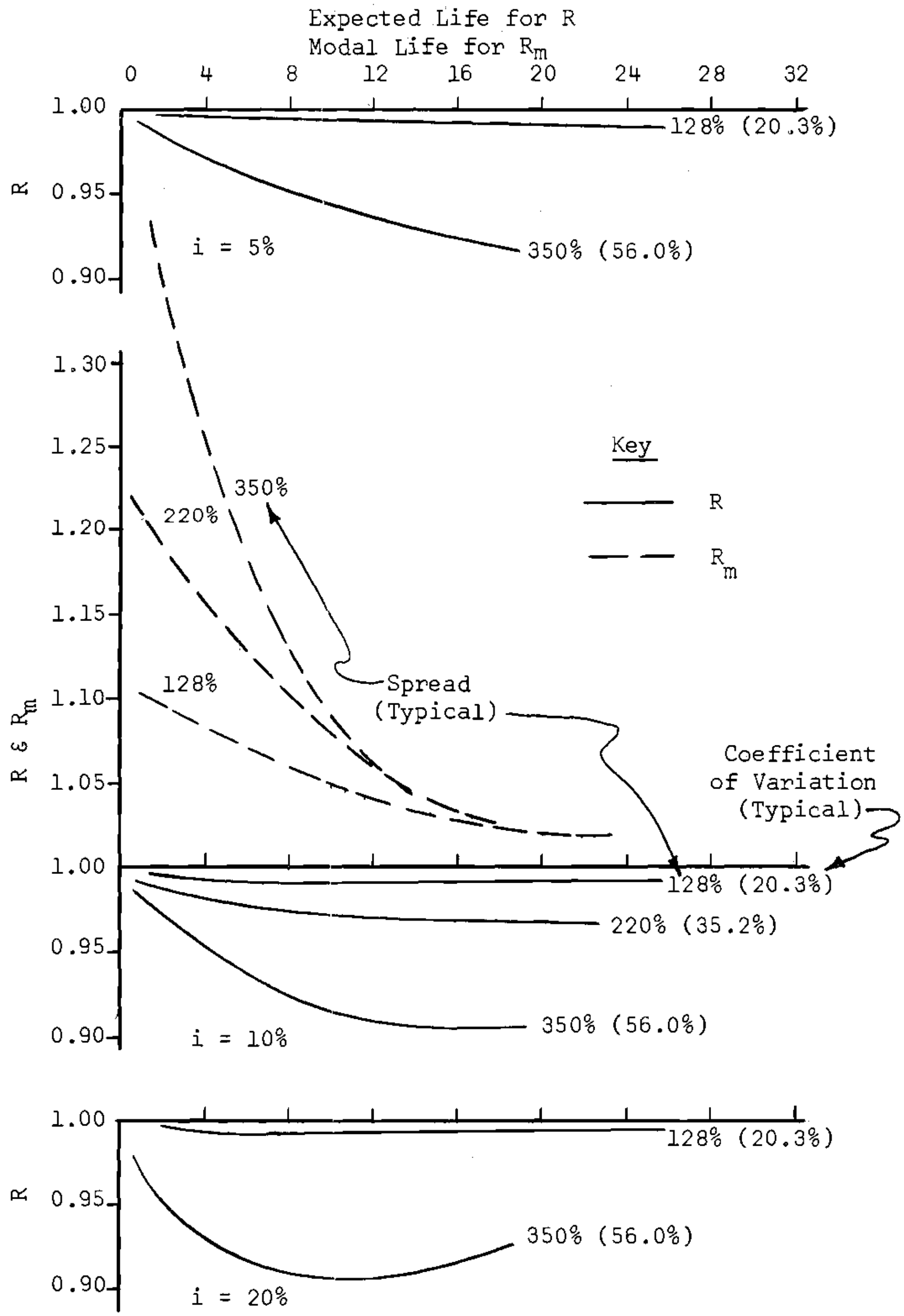


Figure 28. Effect of Beta Distributed Life on R and R_m for Shape Parameters $\alpha = 1$, $\beta = 4$ (Right Skewed)

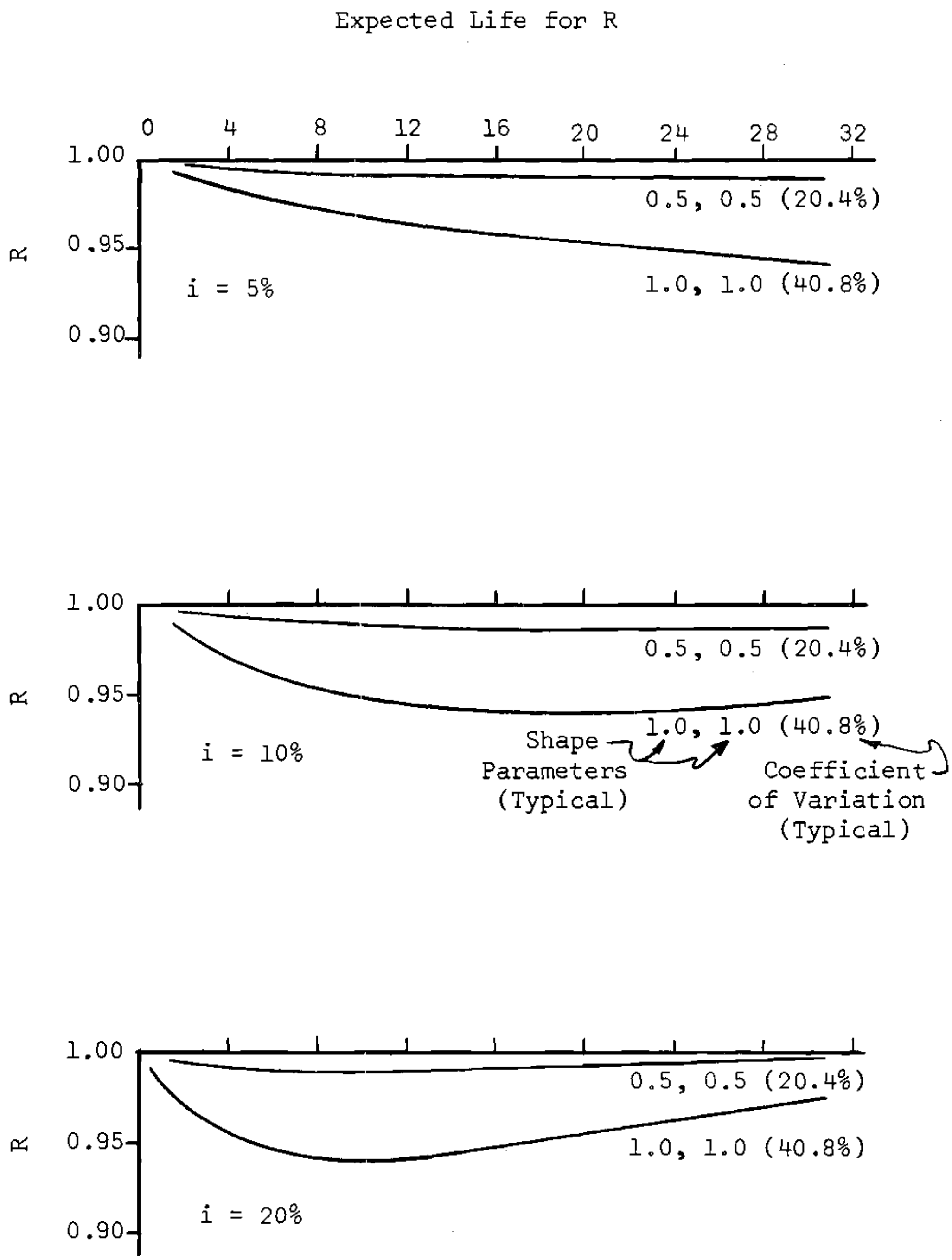


Figure 29. Effect of Triangular Distributed Life on R for Shape Parameters G = 1.0, H = 1.0 and G = 0.5, H = 0.5 (Symmetrical)

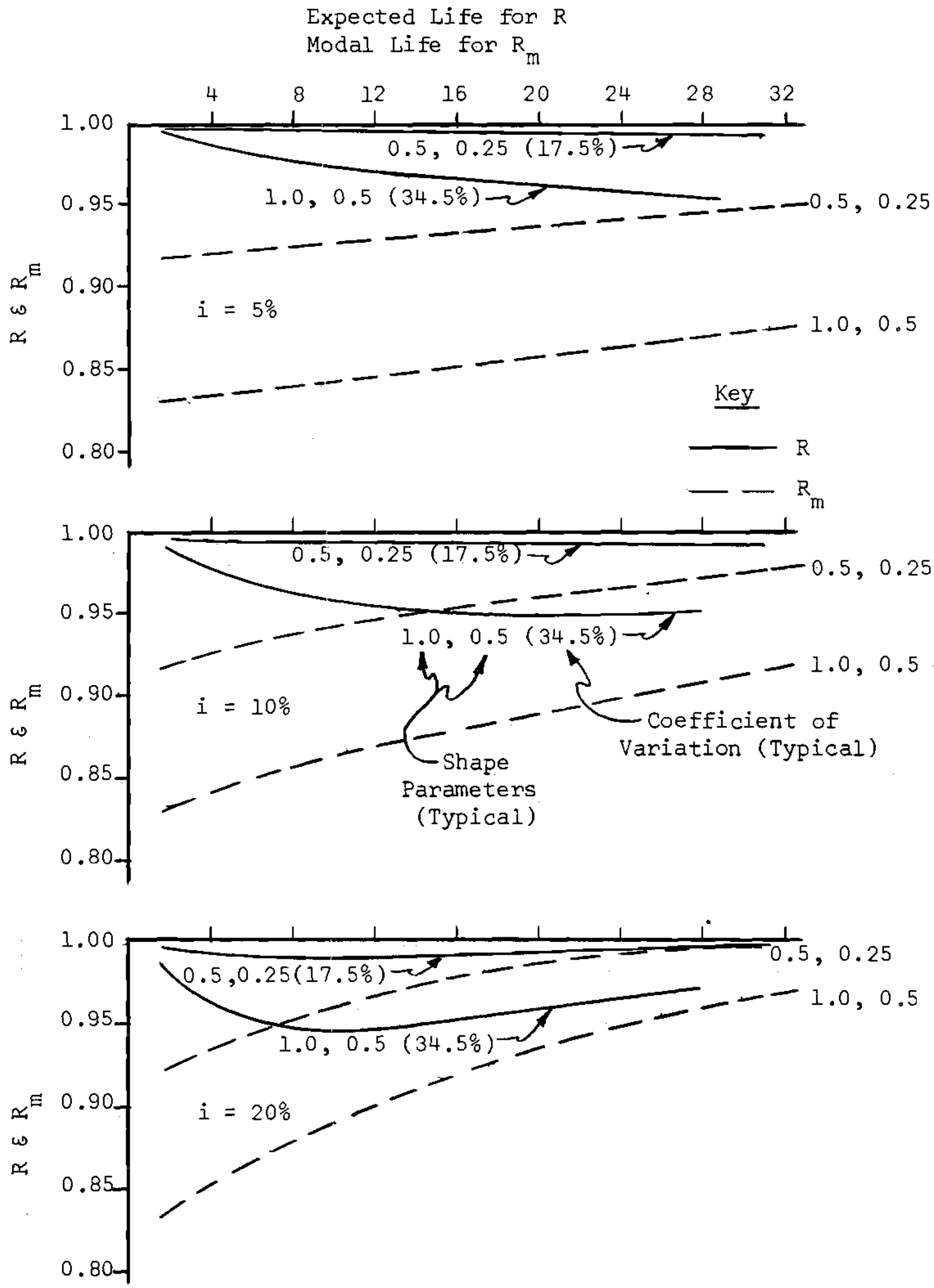


Figure 30. Effect of Triangular Distributed Life on R and R_m for Shape Parameters G = 1.0, H = 0.5 and G = 0.5, H = 0.25 (Left Skewed)

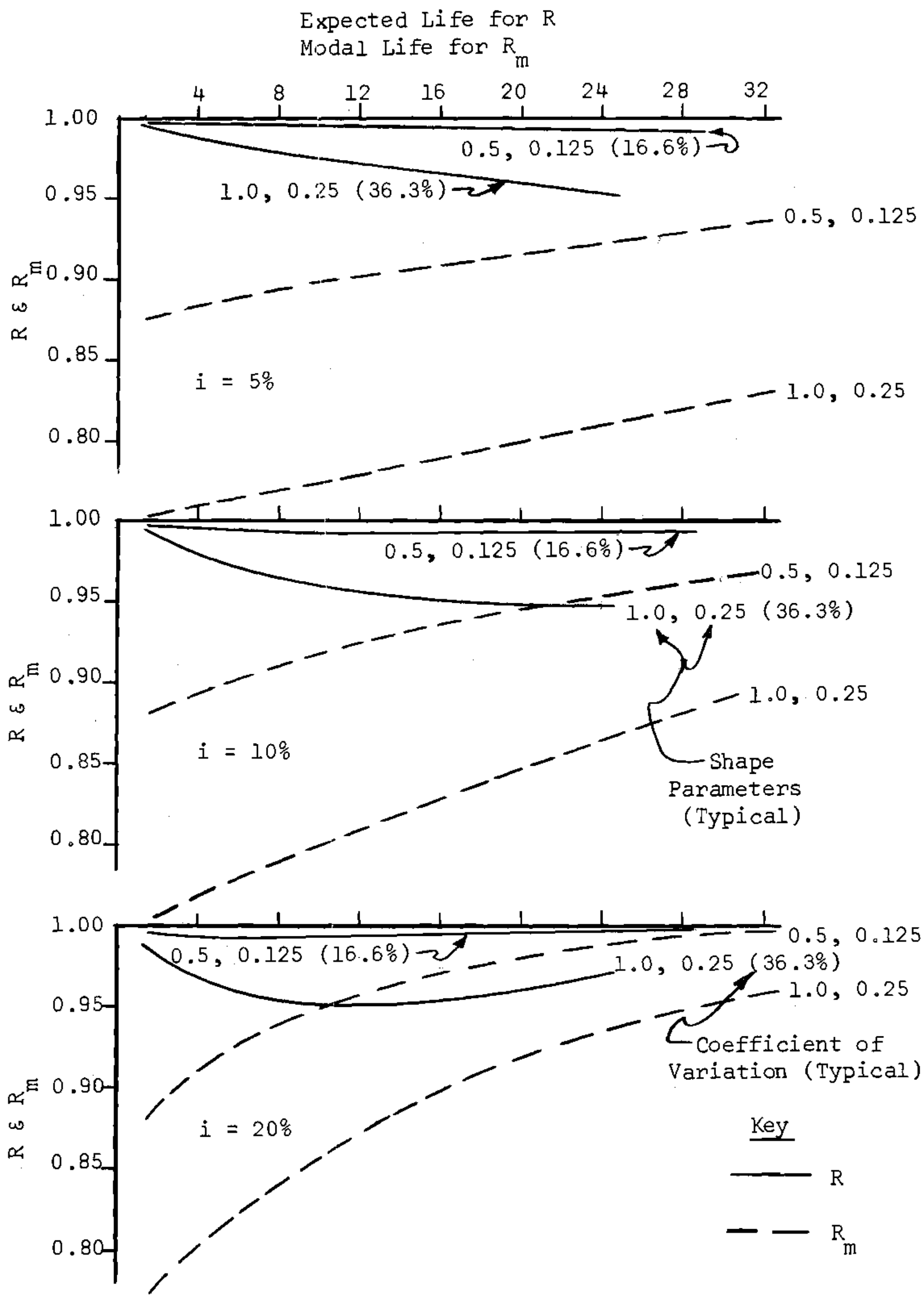


Figure 31. Effect of Triangular Distributed Life on R and R_m for Shape Parameters G = 1.0, H = 0.25 and G = 0.5, H = 0.125 (Left Skewed)

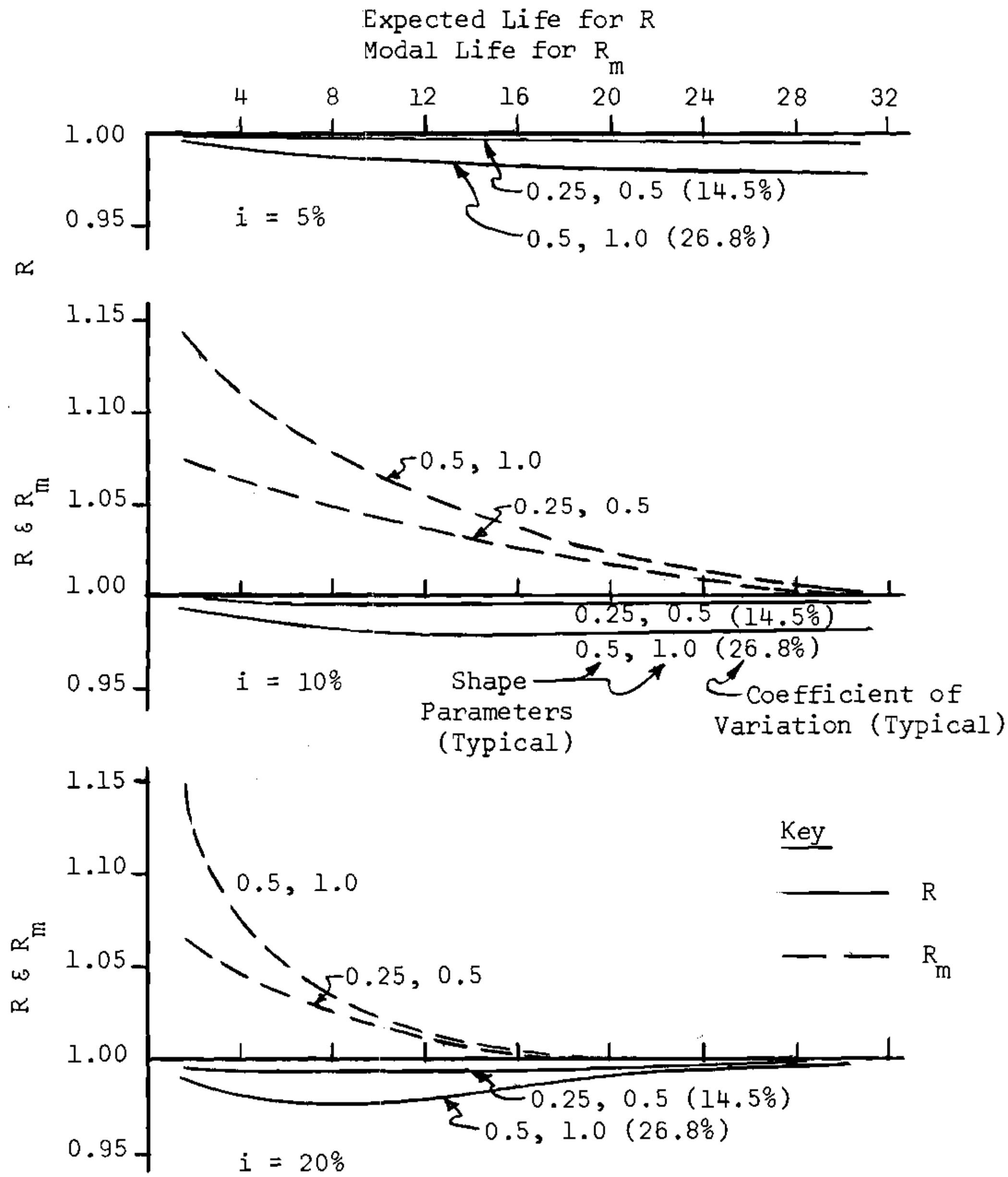


Figure 32. Effect of Triangular Distributed Life on R and R_m for Shape Parameters G = 0.5, H = 1.0 and G = 0.25, H = 0.5 (Right Skewed)

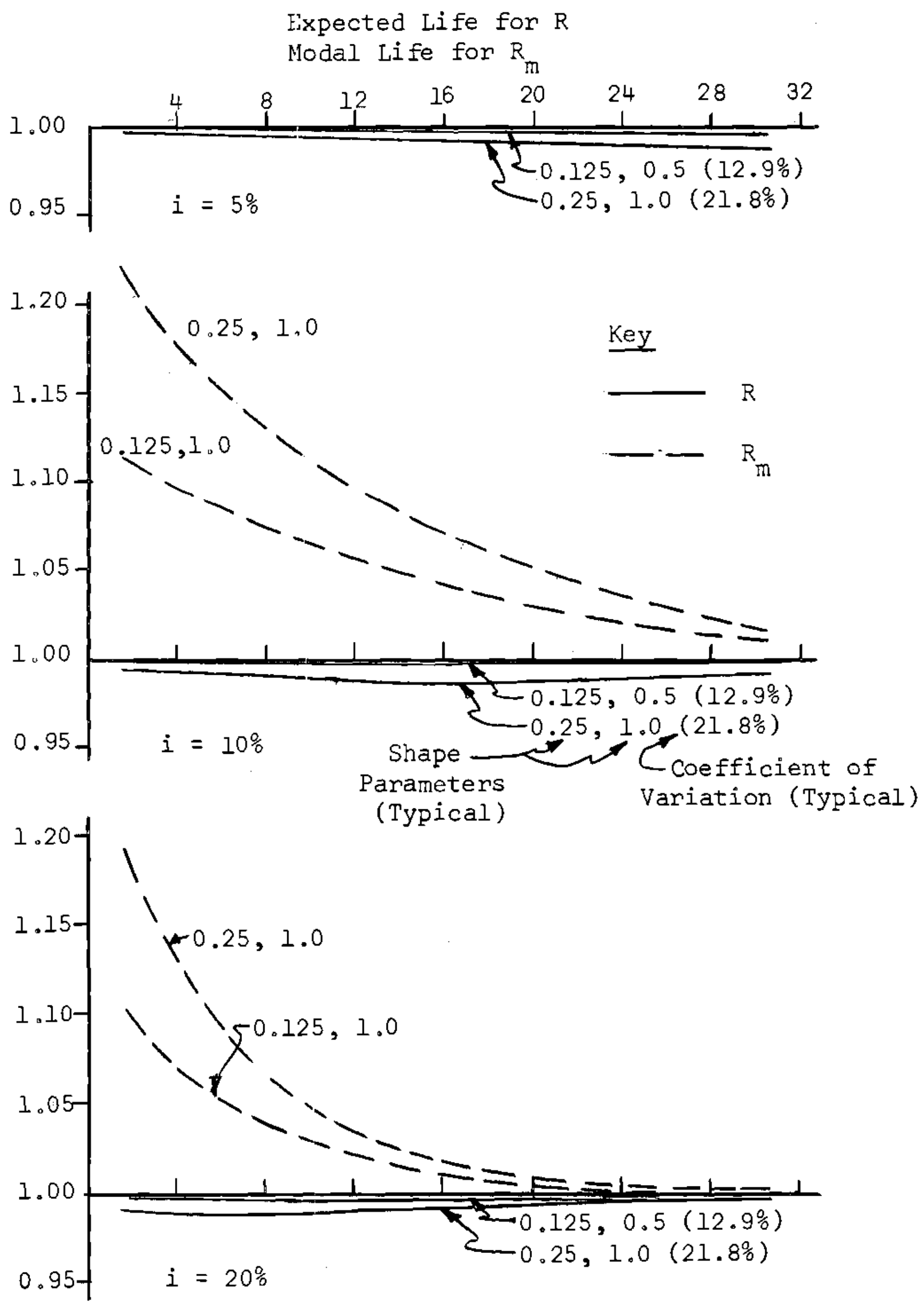


Figure 33. Effect of Triangular Distributed Life on R and R_m for Shape Parameters G = 0.25, H = 0.5 and G = 0.125, H = 0.5 (Right Skewed)

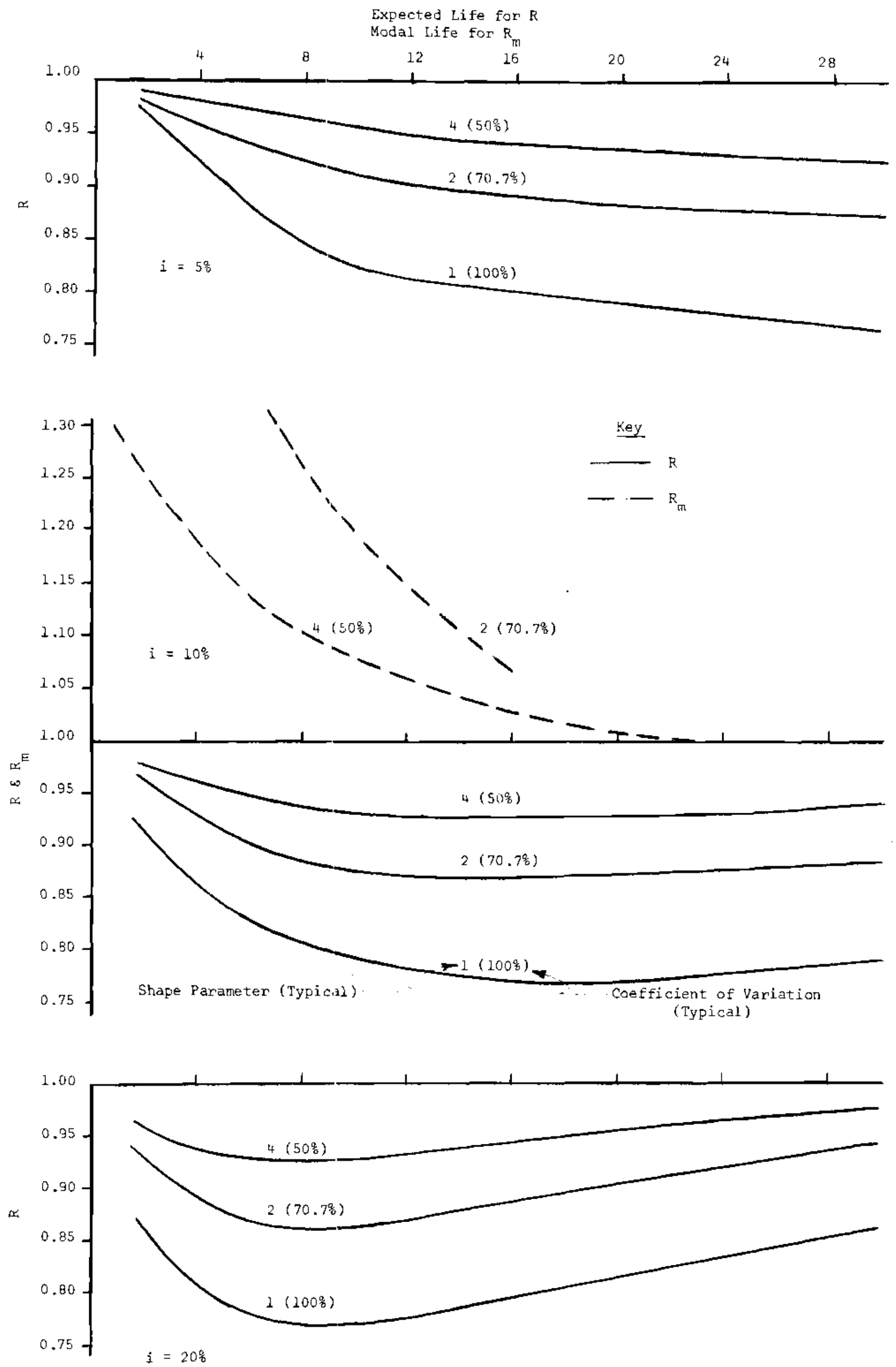


Figure 34. Effect of Gamma Distributed Life on R and R_m

increases to about 0.99 for a coefficient of variation of 20 per cent.

Uniform Distribution (Figure 23). The effect on R of considering a uniformly distributed life is great compared to the effect on R for the other life distributions studied herein. A spread of 200 per cent (coefficient of variation = 57.8 per cent) can result in a minimum R of about 0.88. On the other hand, a spread of 120 per cent (coefficient of variation = 34.9 per cent) results in a minimum R of about 0.96.

Beta Distribution (Figures 24-28). The assumption of a Beta distribution moderately affects R for symmetrical shapes and for right skewed shapes with wide spreads. Table 3 below is a summarization of the approximate R values for i = 10 per cent and at expected lives of 15 years, which is close to the expected life at which R is a minimum for all Beta distribution conditions considered.

Table 3. Typical R Values for Beta Distributed Lives and i = 10 Per Cent

Shape Description	Parameters		Spread Per Cent	Coefficient of Variation (Per Cent)	R at Expected Life of 15 Years
Symmetrical	1	1	200	44.7	.93
			150	33.5	.96
			100	22.4	.98
Left Skewed	2	1	166	33.3	.96
			91	18.2	.99
Left Skewed	4	1	140	22.3	.98
			82	13.1	1.00
Right Skewed	1	2	250	50.0	.92
			176	35.3	.96
			111	22.2	.99
Right Skewed	1	4	350	56.0	.90
			220	35.2	.96
			128	20.3	.99

Triangular Distribution (Figures 29-33). The effect on R of the triangular distribution of life corresponds closely to the Beta distribution for similar amounts of skewness. Table 4 below is a summarization of the approximate R values as determined for i = 10 per cent at expected lives of 15 years, which is close to the life for minimum R for all distribution conditions studied.

Table 4. Typical R Values for Triangular Distributed Lives and i = 10 Per Cent

Shape Description	Parameters G	Parameters H	Coefficient of Variation (Per Cent)	Approximate R at Expected Life of 15 Years
Symmetrical	1.0	1.0	40.8	.95
	0.5	0.5	20.4	.99
Left Skewed	1.0	0.5	34.5	.95
	0.5	0.25	17.5	.99
Left Skewed	1.0	0.25	36.3	.95
	0.5	0.125	16.6	.99
Right Skewed	0.5	1.0	26.8	.98
	0.25	0.50	14.8	.99
Right Skewed	0.25	1.0	21.8	.99
	0.125	0.5	12.9	1.00

Gamma Distribution (Figure 34). The effect of the gamma distribution on R is very pronounced for a shape parameter, α, of 1. When α is 1, R reaches a minimum of about 0.74. For shape parameter of 2, R increases considerably to a minimum of approximately 0.87. For higher shape parameters, which result in more peaked and symmetrical distributions, R comes closer to 1.00. For example, for a shape parameter of 4, the minimum R is about 0.93.

R_m = E(PV)/PV at Modal Life

As in the case of coverage of R_{crm}, rather than pursuing a detailed discussion of quantitative results, general observations will be made, particularly concerning how R_m compares with R.

Beta Distribution (Figures 25-28). For the left skewed cases in Figures 25 and 27, R_m deviates below 1.00 several times more than does R. The deviation of R_m below 1.00 is most pronounced for low expected lives and slowly becomes less for higher expected lives.

For the right skewed cases in Figures 26 and 28, R_m is notably greater than 1.00 for low expected lives, but for increasing expected lives tends to decrease rapidly to the point of becoming somewhat less than 1.00 for high expected lives. In general, the larger the spread of the distribution, the greater the deviation of R_m both above and below 1.00.

Triangular Distribution (Figures 30-33). For the left skewed cases in Figures 30 and 31, R_m deviates below 1.00 several times more than does R. The deviation of R_m below 1.00 is greatest for low expected lives and slowly becomes less for higher expected lives.

For the right skewed cases in Figures 32 and 33, R_m is notably greater than 1.00 for short expected lives but for longer expected lives tends to decrease rapidly and approach 1.00.

Gamma Distribution (Figure 34). R_m has meaning only for a shape parameter, α, greater than 1. For $\alpha = 2$ and $\alpha = 4$, R_m is notably greater than 1.00 for low expected lives, but for increasing expected lives tends to decrease rapidly and approach 1.00.

Effect of Dispersion of Life on Economic Analyses

In this chapter, a rather detailed examination has been made of the relative amount by which consideration of life as a random variable results in an expected capital recovery factor that is greater than the capital recovery factor when computed at the expected life. Also, a similar examination has been made of the relative amount by which consideration of the life distribution results in an expected present value factor that is less than the present value factor when computed at the expected life. This section contains discussion of and examples of the effect of consideration of dispersion of life on economic analyses which combine the use of these two key factors.

If a given project being analyzed consists of only periodic disbursements in addition to those factors which affect capital recovery cost, then consideration of expectations based on the dispersion of life will produce a somewhat cancelling effect. That is, the expected capital recovery cost compared to that cost under assumed certainty will be increased while the expected cost of periodic operating disbursements compared to that cost under assumed certainty will be decreased. If, on the other hand, a given project being analyzed consists of net periodic receipts in addition to those factors which affect capital recovery cost, then consideration of expectations based on the distribution of life will produce a cumulative effect which will decrease the apparent desirability of that project.

For a given project, the relative effect of consideration of dispersion rather than assuming certainty of project life depends upon whether net periodic receipts or only periodic disbursements are considered, the values of R_{cr} or R_{crm} and R or R_m, and the dollar amounts of both capital recovery cost and net periodic receipts or disbursements. When projects are being compared, the relative effect of taking into account project life dispersion depends upon relative differences in those factors for the various projects under consideration.

To illustrate methods and typical results of consideration of life as a random variable in economic analyses, three example problems will be given. In each of the three example problems, an economic analysis will be made using (a) assumed certainty, and (b) the expectation approach considering the life distribution. In each, the analysis measure of merit used is net present value, and the accuracy of each calculation is limited to three significant figures. A short discussion comparing the analysis results will be made for each problem example.

Example Problem I

Given. A project requires an investment of $100,000, and has no salvage value. Project life is normally distributed with E(T) of 5 years and a σ of 1.5 years. Periodic net cash receipts are $30,000 annually. It is expected that this project will be replaced one time with an identical project and that the total length of time that the series of projects will last is normally distributed with a mean of ten years and a standard deviation of two years. Interest is 10 per cent. It is desired to determine if the project is acceptable.

Solution Assuming Certainty.

$$CRC(\$) = \$100{,}000\left(\frac{i}{1-e^{-iT}}\right) = \$100{,}000\left(\frac{0.1}{1-e^{-0.1(5)}}\right) = \$25{,}400.$$

Net Annual Receipts After CRC($) = $30,000-$25,400 = $4,600.

$$PV(\$)\ \{\text{for series}\} = \$4{,}600\left(\frac{1-e^{-iT}}{i}\right) = \$4{,}600\left(\frac{1-e^{-0.1(10)}}{0.1}\right) = \underline{\underline{\$29{,}100}}.$$

Solution Considering Life Dispersion.

From Figure 6; $R_{cr}(\text{for } i=10\%, E(T)=5 \text{ years}, \frac{\sigma}{E(T)} = \frac{1.5}{5} = 30\%) = 1.07.$

$$R_{cr} = \frac{E(CRF)}{\text{CRF at Exp. Life}}\ ;\ \therefore\ E(CRF) = (\$1.07)\cdot(0.254) = 0.272.$$

$E(CRC(\$)) = (P)\cdot E(CRF) = \$100{,}000(0.272) = \$27{,}200.$

Net Annual Receipts After CRC($) = $30,000-$27,200 = $2,800.

From Figure 22; $R(\text{for } i=10\%, E(T) = 10 \text{ years}, \frac{\sigma}{E(T)} = \frac{2}{10} = 20\%) = 0.99.$

$$R = \frac{E(PV)}{\text{PV at Exp. Life}}\ ;\ E(PV) = 0.99(6.32) = 6.25.$$

$E(PV(\$)) = \$2{,}800\ (6.25) = \$17{,}500.$

When assuming certainty, the project is acceptable (i.e., positive present value). When considering dispersion of life, the project is still acceptable, though markedly less acceptable than when assuming certainty.

Example Problem II

Given. A project requires an investment of $200,000 and has no salvage value. Project life is uniformly distributed with a minimum life of two years and a maximum of 18 years (thus, E(T) = 10 years, and spread = 160 per cent). Annual net cash receipts are $37,000. Interest is 10 per cent. It is desired to determine if the project is acceptable.

Solution Assuming Certainty.

$$CRC(\$) = \$200,000\left(\frac{i}{1-e^{-iT}}\right) = \$200,000\left(\frac{0.10}{1-e^{-0.10(10)}}\right) = \$31,600.$$

Net Annual Receipts After CRC($) = $37,000-$31,600 = $5,400.

$$PV(\$) = \$5,400\left(\frac{1-e^{-iT}}{i}\right) = \$5,400\left(\frac{1-e^{-0.10(10)}}{0.10}\right) = \underline{\underline{\$34,100}}.$$

Solution Considering Life Dispersion.

From Figure 7; R_{cr}(for i=10%,E(T)=10 years, spread = 160%) = 1.23.

$$R_{cr} = \frac{E(CRF)}{\text{CRF at Exp. Life}};\ E(CRF) = (0.158)\cdot(1.23) = .194.$$

E(CRC($)) = (P)·E(CRF) = $200,000(0.194) = $38,800.

Net Annual Receipts After CRC($) = $37,000-$38,800 = -$1,800.

From Figure 23, R(for i=10%,E(T)=10 years, spread = 160%) = 0.94.

$$R = \frac{E(PV)}{\text{PV at Exp. Life}};\quad E(PV) = 0.94(6.32) = 5.94.$$

$$E(PV(\$)) = -\ \$1,800(5.94) = \underline{\underline{-\$10,700}}.$$

When assuming certainty, the project is acceptable (positive present value). When considering dispersion of life, the project is not acceptable (negative E(PV($)). Hence, the apparent desirability of the project is reversed when taking into account dispersion of life.

Example Problem III

Given. The same two projects as given in Problems I and II are to be compared. Assume that the two projects are independent and mutually exclusive. It is desired to choose which project is best.

Solution. The following table is given to summarize the relevant results:

Table 5. Comparison of Analysis Results Using Assumed Certainty Versus Considering Distribution of Life

Mode of Analysis	Results in PV($) Project in Problem I	Project In Problem II
(a) Assumed Certainty	$29,100	$34,100
(b) Expectation Considering Dispersion of Life	17,500	-10,700

By inspection of Table 5, it can be seen that if the mode of analysis is assumed certainty, then the project in Problem II would be the choice (higher PV($)). On the other hand, if the mode of analysis considers the life distribution, the project in Problem I would be the choice. This last example problem illustrates a case where it could be worthwhile for the analyst to consider the life distributions rather than to base the analysis on assumed certain project lives.

Exploration of the effect of variation of only project life is now complete. The next chapter is devoted to the consideration of dispersion of other important elements in addition to project life.

CHAPTER IV

THE EFFECT OF VARIATION OF MULTIPLE ELEMENTS IN ECONOMIC ANALYSES

Introduction

This chapter is several steps closer to the needs inherent in practice than is Chapter III because it considers the effect of variation of combinations of several major elements that affect economic analyses. The major elements for which the effect of dispersion is to be taken into account are the amounts of the investment, project life, annual net cash receipts or disbursements, and salvage value. Because of the number of elements considered together, various probability distributions for each of the elements will not be considered. Sensitivity studies will be made to show the relative change in an economic analysis criterion caused by changes in major elements comprising the analysis. Consideration will primarily be made of the mean and variance of each of the elements and their effect on the mean and variance of the measure of merit.

The measure of merit which will be used primarily is the present value (present worth). The present value of one life cycle of a project is given by the general relation:

$$PV(\$) = \sum_{t=0}^{T} X_t e^{-t} \qquad (19)$$

where X_t is the cash flow for the t^{th} year, assumed to occur at the end of each year. A cash inflow will have a positive coefficient while an outflow will have a negative coefficient. Equation (19) can be broken down into the form in Equation (20) below which shows the major elements for which variation is to be taken into account.

$$PV(\$) = P + \sum_{t=1}^{T} D_t e^{-it} + Se^{-iT} \tag{20}$$

where D_t is the net cash receipts or disbursements for the t^{th} year, and all other symbols have previously been defined.

If D_t is assumed to occur at a constant rate and continuously within each year and interest is assumed to be compounded continuously, then D_t can be replaced with D and $\sum_{t=1}^{T} e^{-it}$ becomes $\int_0^T e^{-it}dt$, which equals $(1-e^{-iT})/i$. The result is the simplified expression for finding the present value of one life cycle of a project shown below:

$$PV(\$) = P + D((1-e^{-iT})/i) + Se^{-iT} . \tag{21}$$

The assumed certainty mode of study as traditionally practiced calls for evaluating a given project according to Equation (21) using expected or mean values for each of the elements in the right hand side of the equation so as to arrive at "the" value for PV(\$). The criterion for desirability for a single project for which revenues are known is that the project is "acceptable" only if the PV(\$) is greater than zero, which means that the project would increase the expected total wealth of the firm more than investing the same money elsewhere at the rate of return, i. The criterion for desirability normally used when comparing mutually exclusive alternatives over a given study period is that the alternative with the most positive value of PV($) would be preferred.

In a given situation, any or all of the elements that comprise Equation (21) for PV($) are likely to be random variables and may be interrelated. Hence PV($) is likely to be a random variable. Work which follows later in this chapter will demonstrate a means of approximating and interpreting properties of the random variable PV($). The next section is a limited study of the sensitivity of PV($).

Sensitivity to Changes in Major Elements in Present Value Analyses

This study of sensitivity involves a determination of the relative effect on an economic analysis criterion, PV($), caused by changes in each of the elements that affect that analysis. The idea of using this type of study is suggested by William Morris (65-pp.256-257). Some elements in an analysis may change by substantial amounts without appreciably affecting the analysis result. However, there may be other elements for which a nominal change will affect the analysis result considerably. The analysis result would be said to be highly sensitive to these latter type elements. It is worthwhile to know the sensitivity to each of the elements so that the analyst can be prepared to consider most carefully the risk of changes in those elements which have the greatest effect on the results of the analysis.

This study of sensitivity of PV($) is pursued by taking partial derivatives of Equation (21) with respect to each of the major elements of interest and plotting the results for a wide range of conditions. The partial derivatives are:

$$\frac{\partial PV(\$)}{\partial P} = 1,$$

$$\frac{\partial PV(\$)}{\partial D} = \frac{1-e^{-iT}}{i},$$

$$\frac{\partial PV(\$)}{\partial i} = D\left[\frac{(iT+1)e^{-iT} - 1}{i^2}\right] - STe^{-iT}, \tag{22}$$

$$\frac{\partial PV(\$)}{\partial T} = De^{-iT} - Sie^{-iT}, \text{ and}$$

$$\frac{\partial PV(\$)}{\partial S} = e^{-iT}.$$

Figures 35-38 show graphically the above partial derivatives for use in sensitivity studies over a range of 2 to 30 years and for interest rates of 5, 10, and 20 per cent. Since $\partial PV(\$)/\partial P$ is a constant, it is not shown graphically.

It is worthy of note that the sensitivity of PV($) to a change in cash flow for any given year, D_t, is the same as the sensitivity of the PV($) to a change in salvage value for that year. This is shown in Figure 38.

Examination of Figures 35-38 reveals that in general the higher the interest rate, the lower the sensitivity to changes in the major elements. Another relationship to note in Figures 35-38 is that the higher the life, the higher the sensitivity to changes in annual receipts

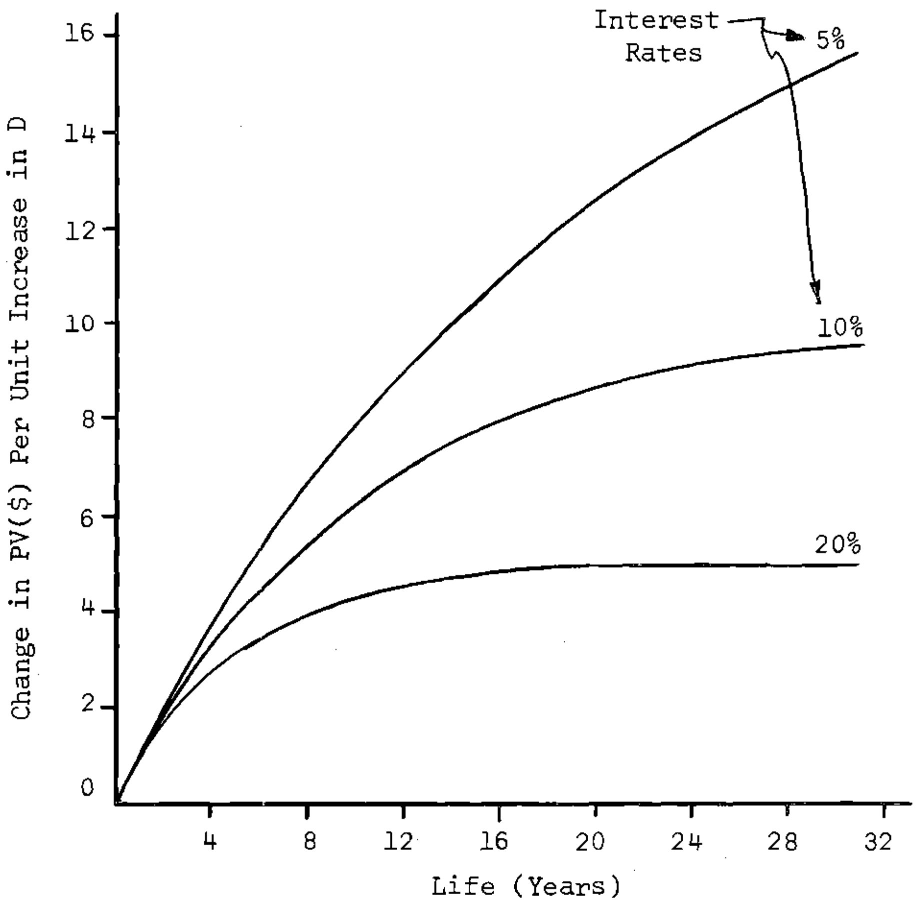


Figure 35. Rate of Change in Present Value with Change in Annual Receipts or Disbursements

$$\frac{\delta PV(\$)}{\delta D} = \frac{1-e^{-iT}}{i}$$

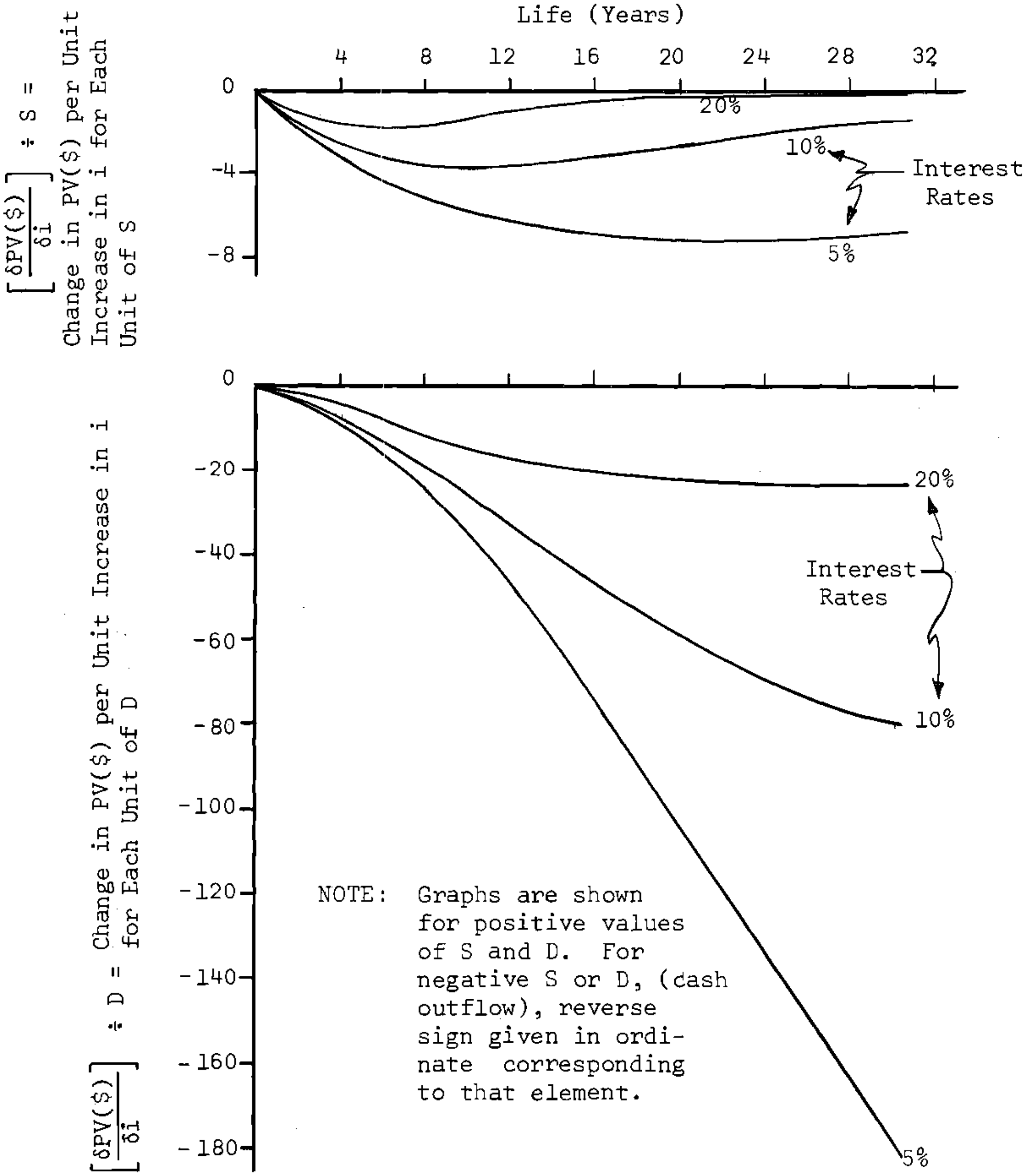


Figure 36. Rate of Change in Present Value with Change in Interest Rate

$$\frac{\delta PV(\$)}{\delta i} = D\left[\frac{(iT+1)e^{-iT}-1}{i^2}\right] - STe^{-iT}$$

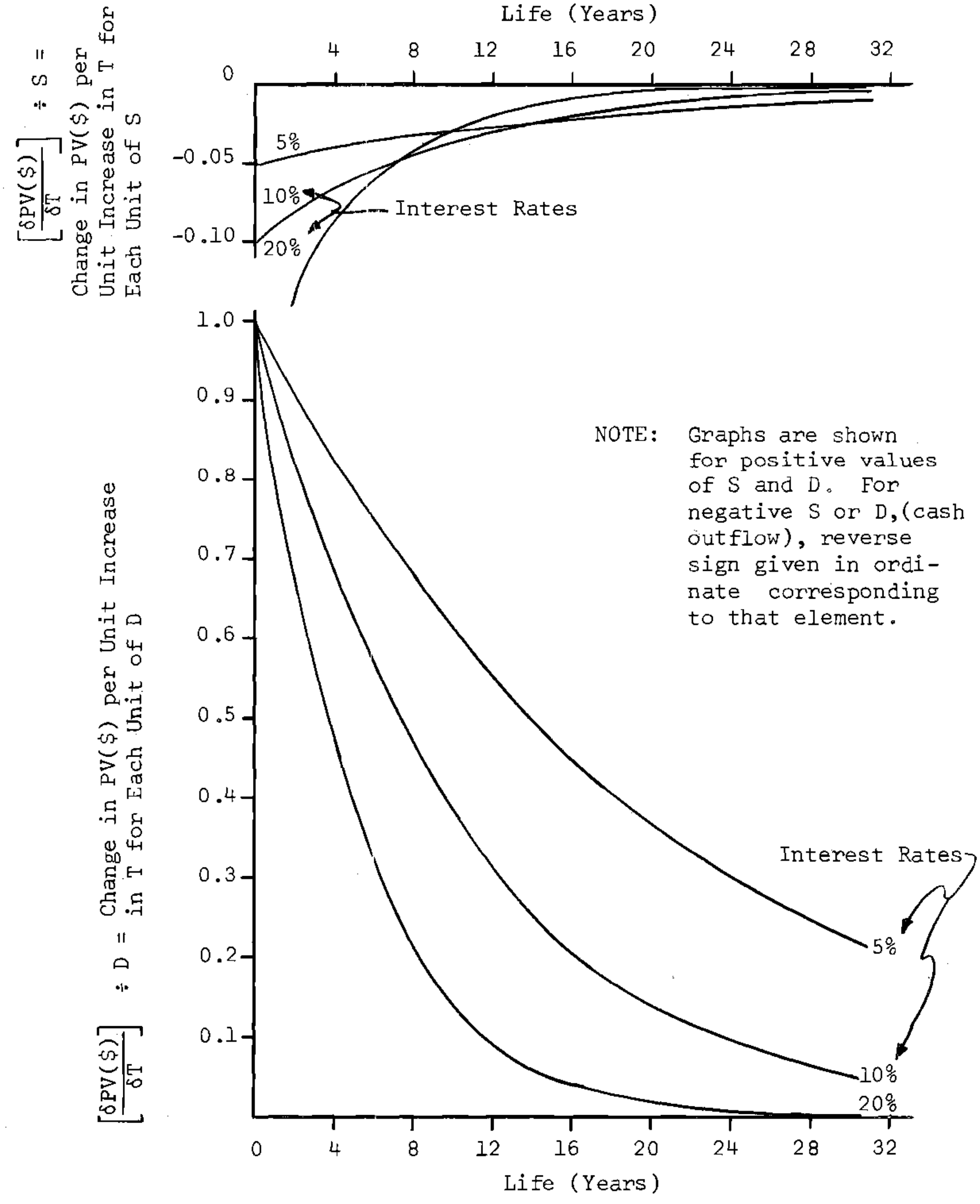


Figure 37. Rate of Change in Present Value with Change in Life of Project

$$\frac{\delta PV(\$)}{\delta T} = De^{-iT} - Sie^{-iT}$$

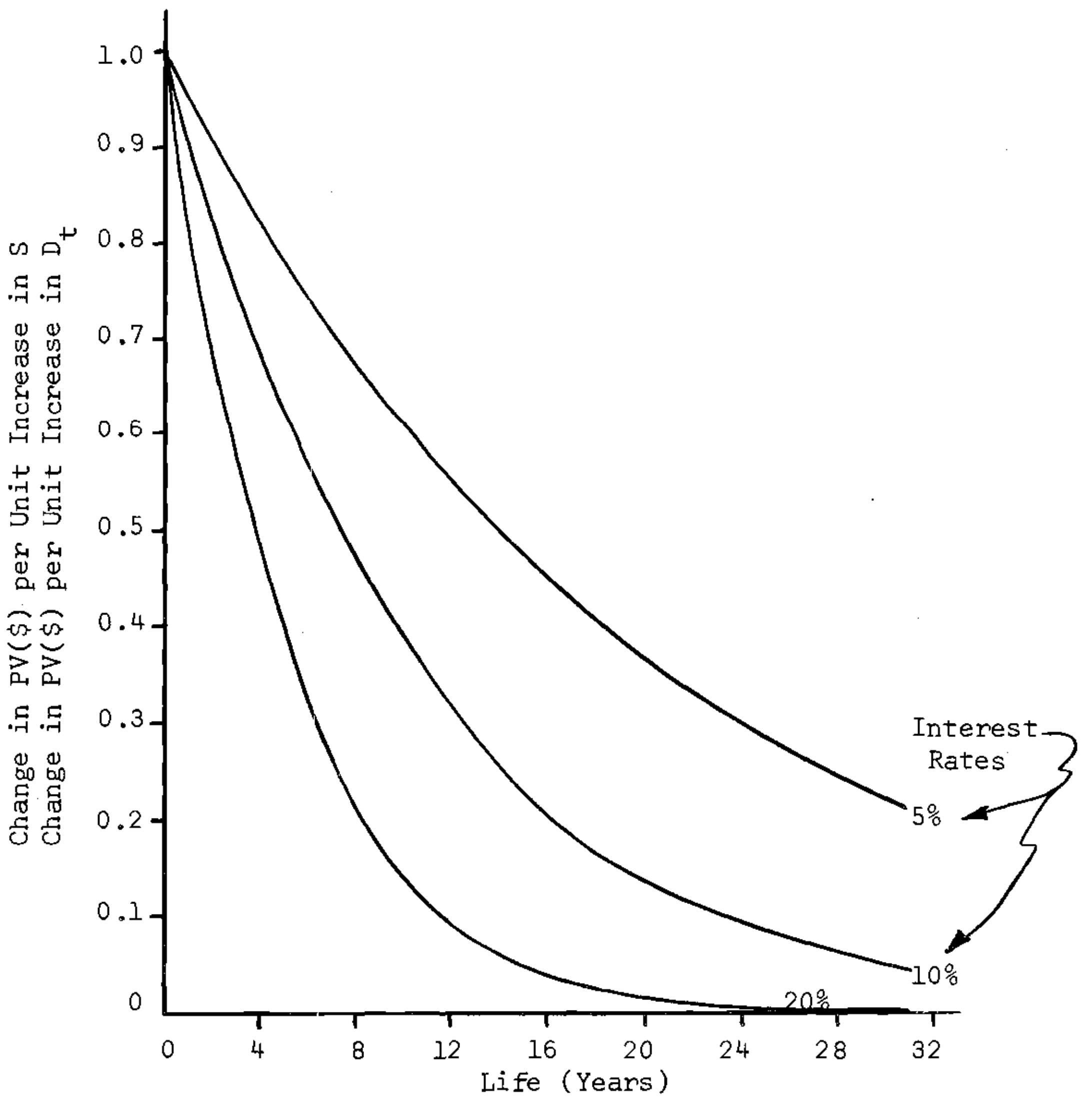


Figure 38. Rate of Change in Present Value with Change in Salvage Value or Compared to Change in Cash Flow for any Particular Year

$$\frac{\delta PV(\$)}{\delta S} = e^{-iT}$$

$$\frac{\delta PV(\$)}{\delta D_t} = e^{-it}$$

or disbursements and to changes in the interest rate. On the other hand, the higher the life, the lower the sensitivity to changes in either the life or the salvage value.

Approximation of Expected Value and Variance of PV($) for a Project

A simplified relation for finding PV($) for a life cycle of a project was shown in Equation (21). It has already been noted that each of the elements that comprise this relation may well be subject to variation. The purpose of this section is to develop means for obtaining workable approximations of the mean and variance of PV($) when taking into account the dispersion of the individual elements. It should be noted again that the interest rate, i, is assumed to be a constant.

Expected Value of PV($)

If S and D are each independent of T, the expected value of Equation (21) may be shown to be:

$$E(PV(\$)) = E(P) + E(D)\cdot E((1-e^{-iT})/i) + E(S)\cdot E(e^{-iT}) \qquad (23)$$

$E((1-e^{-iT})/i)$ may be obtained conveniently for a wide range of distributions of T through the use of Figures 22-34 shown in Chapter III. Those figures are designed to show R = E(PV)/PV at Expected Life. The quantity denoted as E(PV) in Chapter III is $E((1-e^{-iT})/i)$. Thus,

$$E((1-e^{-iT})/i) = E(PV) = (R)\cdot(PV \text{ at Expected Life}). \qquad (24)$$

Similar graphs have not been developed for convenience in finding $E(e^{-iT})$ for various distributions of T. However, $E(e^{-iT})$ can be calculated by the relation:

$$E(e^{-iT}) = \int_A^Z e^{-iT} \cdot f(T)dt \tag{25}$$

For practical applications where the individual elements can be assumed to be mutually independent, an approximation of E(PV($)) that may well be sufficiently close can be obtained through evaluation of a Taylor series expansion of PV($) about the mean of each element.[5] If only the first two terms of the expansion are concerned, the resulting E(PV($)) can be obtained by merely evaluating Equation (21) at the expected value of each of the individual elements.

<u>Variance of PV($)</u>

If the various elements comprising PV($) are mutually independent, the variance of the distribution of PV($) may be conveniently approximated by the use of the first two terms of the Taylor series expansion of PV($) as:

$$V(PV(\$)) \approx \left(\frac{\partial PV(\$)}{\partial P}\right)^2 \cdot V(P) + \left(\frac{\partial PV(\$)}{\partial D}\right)^2 \cdot V(D) + \left(\frac{\partial PV(\$)}{\partial T}\right)^2 \cdot V(T) + \left(\frac{\partial PV(\$)}{\partial S}\right)^2 \cdot V(S). \tag{26}$$

Note that the above formula does not consider variance in the interest rate. If it is desired to consider variance in interest rate

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5. See Appendix E for discussion of Taylor Series expansion.

or in any other element, another term similar to the four terms above should be added for each additional element considered. The partial derivatives for use in Equation (26) are shown in Equations (22). As a means of saving manual computations, Figures 39-41 have been prepared to allow the practitioner to easily obtain $(\partial PV(\$)/\partial T)^2$, $(\partial PV(\$)/\partial S)^2$, and $(\partial PV(\$)/\partial D)^2$, respectively, for a wide range of conditions. These figures can also be used as an aid to visually examining the relative effect of different conditions on the various terms which contribute to V(PV($)).

Since the variance of D is often a major contributor to the total V(PV($)) for a project, the next section will contain detailed means for the calculation of the contribution to V(PV($)) by D_t. The detailed means consider the variance of D_t year-by-year, as well as the correlation between the cash flows for each year. This contribution to total V(PV($)), which can replace $(\partial PV(\$)/\partial D)^2 \cdot V(D)$ in Equation (26), will be called $V_D(PV(\$))$.

Detailed Means for Calculation of $V_D(PV(\$))$. Let σ_s and σ_t be the standard deviation of the cash flow for years s and t, respectively. Let ρ_{st} be the coefficient of correlation relating the correlation of cash flow for year s to year t. The covariance of cash flows for years s and t, σ_{st}, can be calculated by the following relation:

$$\sigma_{st} = \rho_{st}\, \sigma_s\, \sigma_t \ . \tag{27}$$

The matrix of variances and covariances relating cash flows for each pair of years during the life of a project can be shown by the matrix in Figure 42.

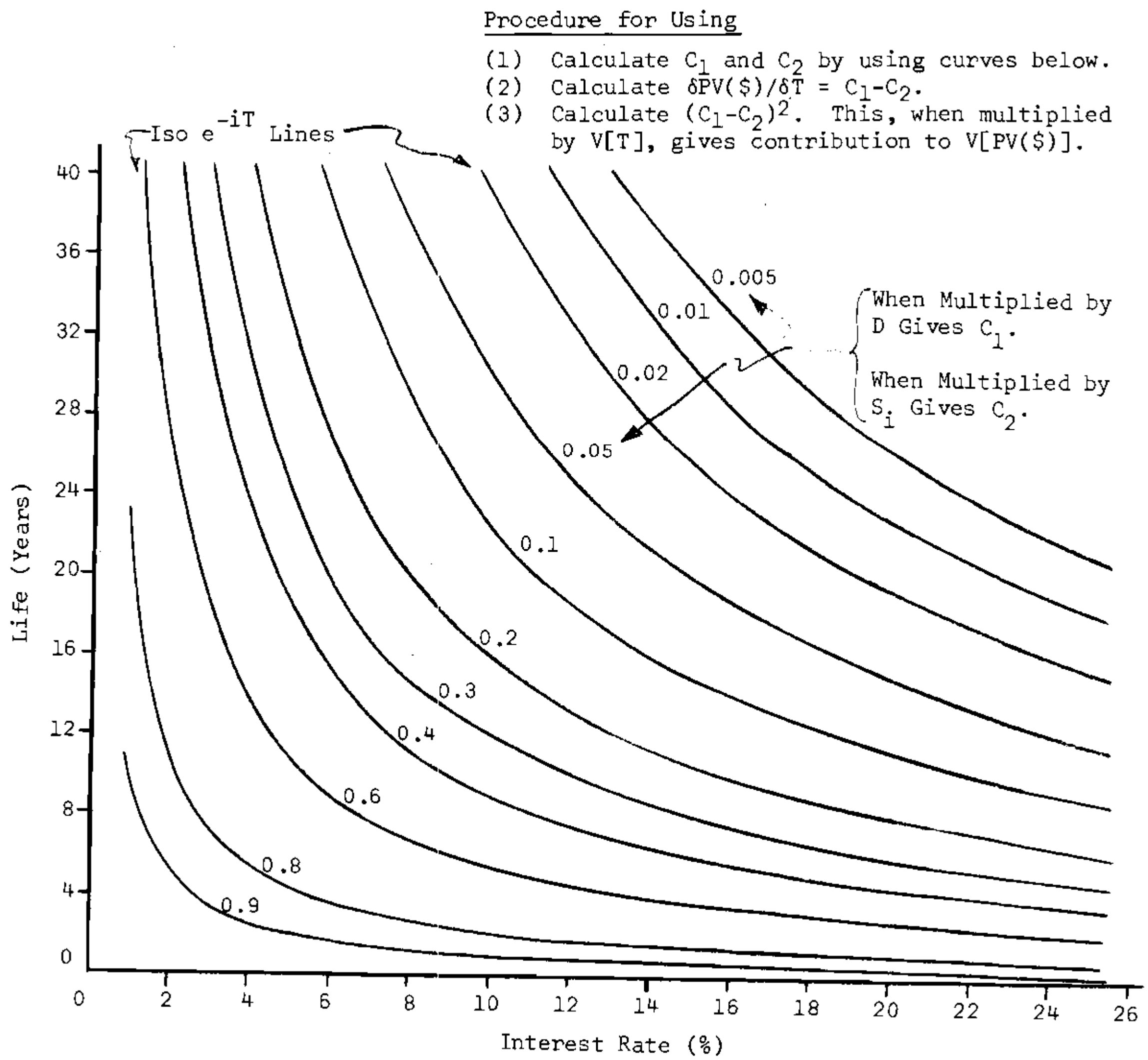


Figure 39. Contribution to Variance of PV($) Due to Variance of T

$$(\delta PV(\$)/\delta T)^2 = (De^{-iT} - Sie^{-iT})^2$$

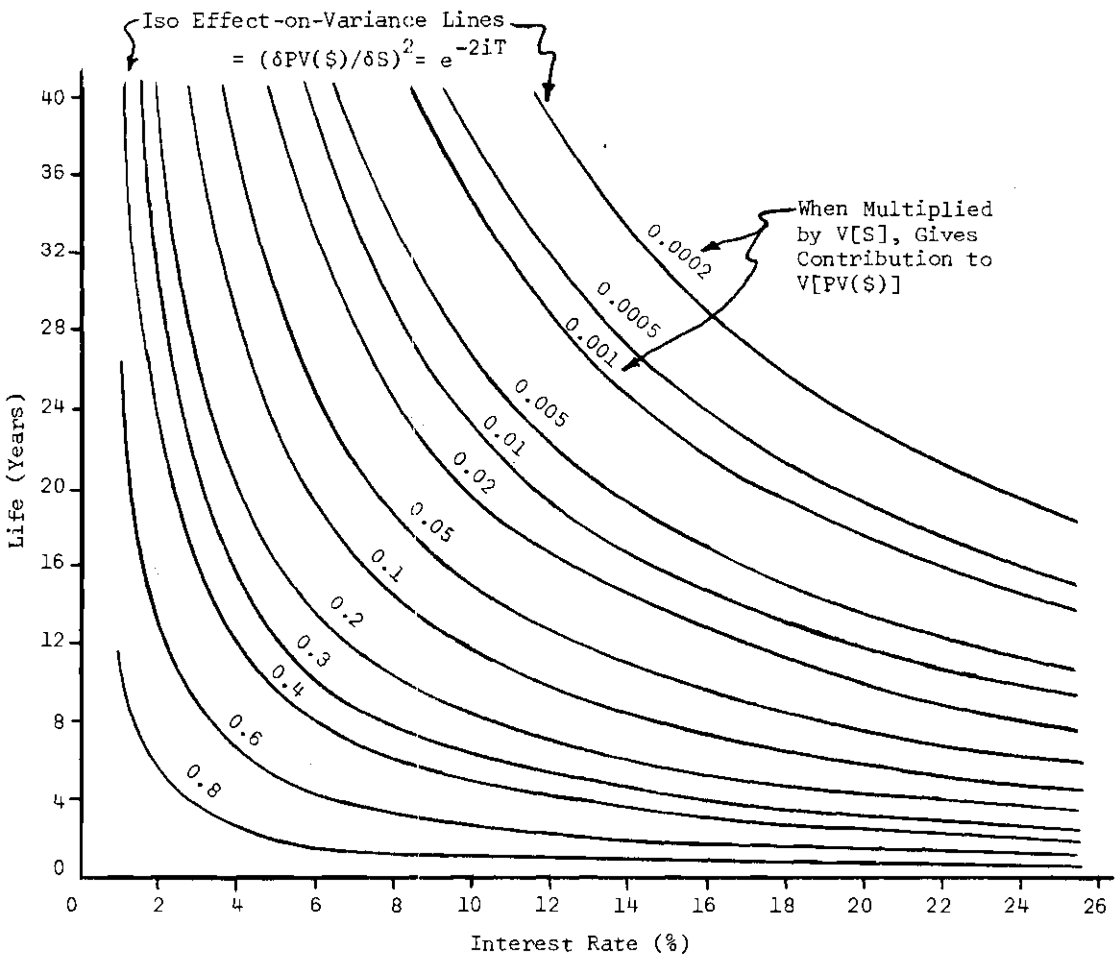


Figure 40. Contribution to Variance of PV($) Due to Variance of S

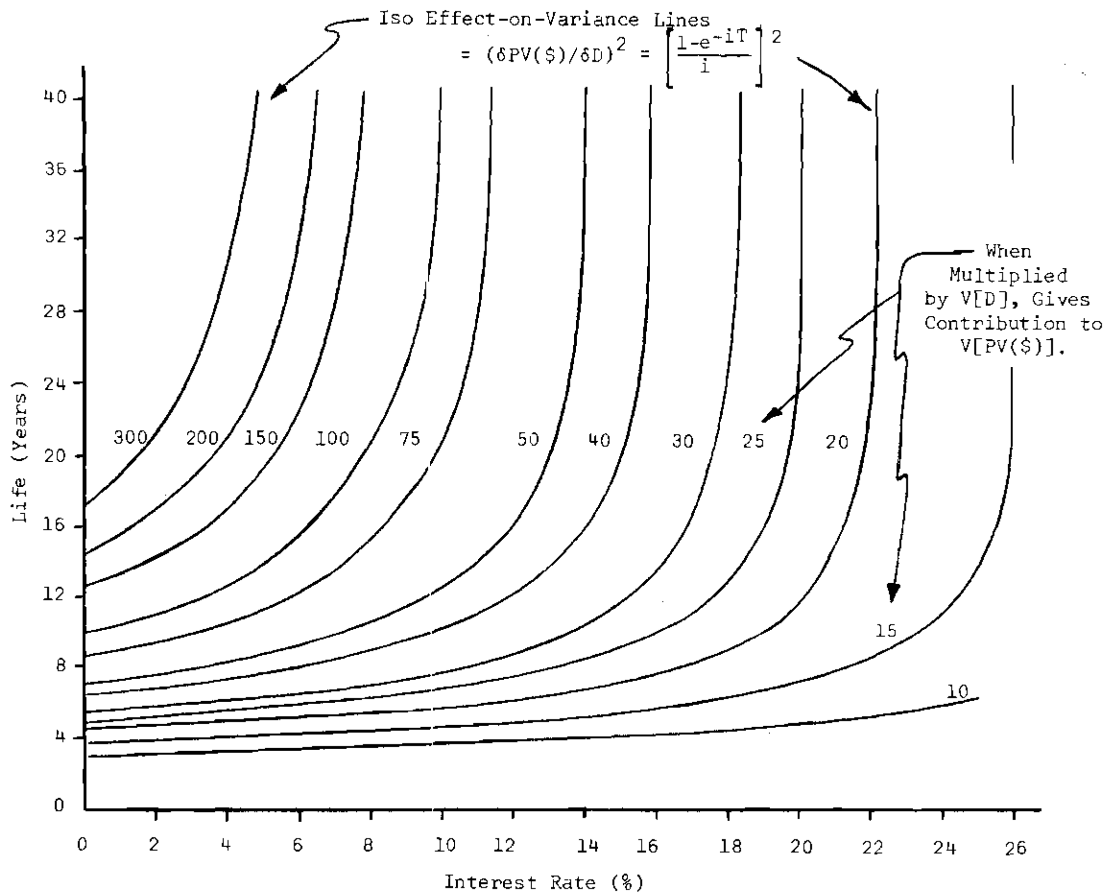


Figure 41. Contribution to Variance of PV($) Due to Variance of D

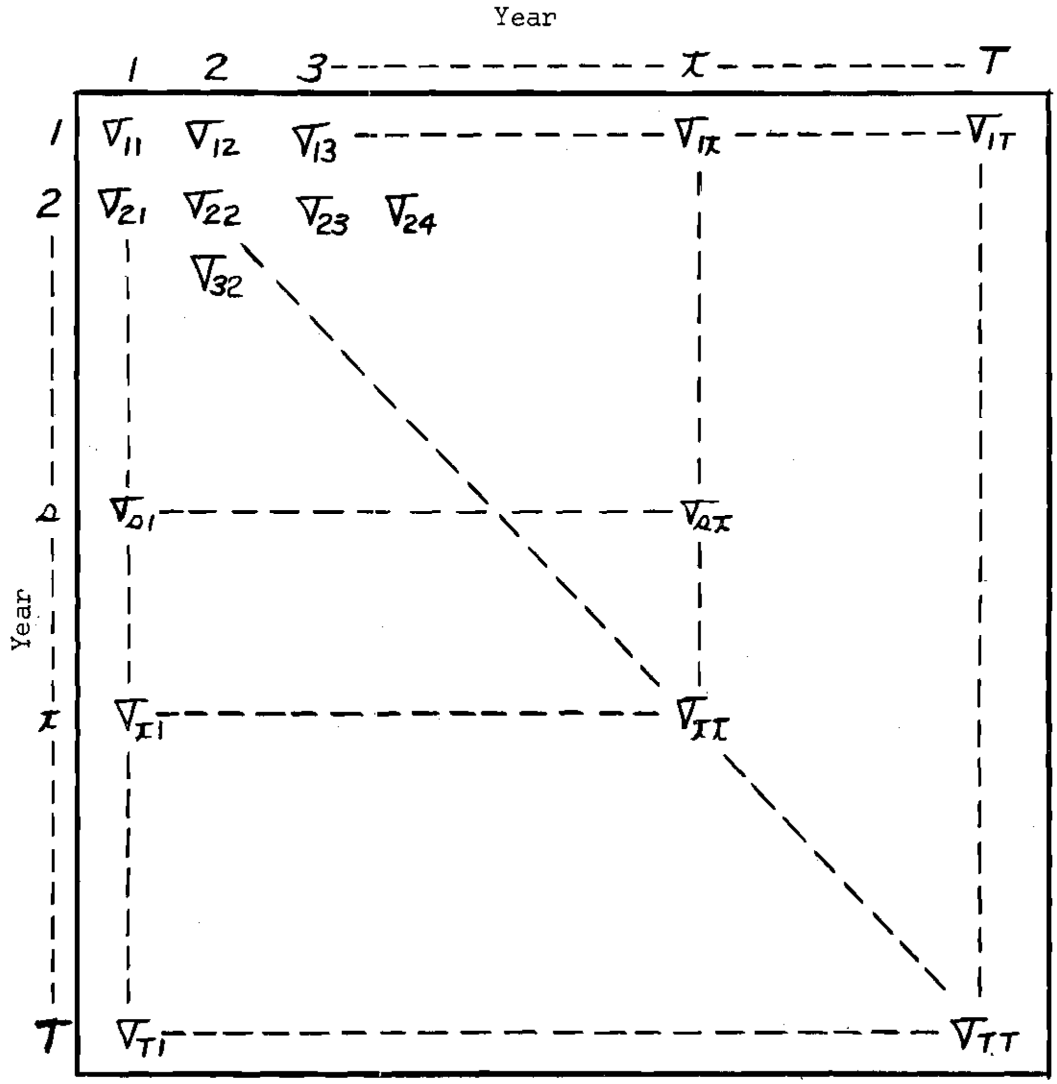


Figure 42. Matrix of Variances and Covariances of Cash Flows for all Years

The variance of the present worth of the cash flows, $V_D(PV(\$))$, can be calculated from the above matrix by summing the discounted covariances. Thus,

$$V_D(PV(\$)) = \sum_{\substack{\text{all } s,\\ t \text{ pairs}}} \sigma_{st}\, e^{-i(s+t)}. \tag{28}$$

This variance can be separated into two components as:

$$V_D(PV(\$)) = \sum_{\substack{\text{all pairs}\\ \text{such that}\\ s = t}} \sigma_{st}\, e^{-i(2t)} + \sum_{\substack{\text{all pairs}\\ \text{such that}\\ s \neq t}} \sigma_{st}\, e^{-i(s+t)}. \tag{29}$$

Since $\sigma_{st} = \sigma_s^{\ 2} = \sigma_t^{\ 2}$ where $s = t$, and since the matrix in Figure 42 is symmetrical on each side of the main diagonal,

$$V_D(PV(\$)) = \sum_{t=1}^{T} \sigma_t^{\ 2}\, e^{-2it} + 2 \sum_{\substack{\text{all pairs}\\ \text{such that}\\ s > t}} \sigma_{st}\, e^{-i(s+t)}. \tag{30}$$

σ_{st} in the right hand side of Equation (30) can be replaced by $\rho_{st}\ \sigma_s\ \sigma_t$ so as to use coefficients of correlation to reflect degrees of dependence in the cash flows for each pair of years. Thus,

$$V_D(PV(\$)) = \sum_{t=1}^{T} \sigma_t^{\ 2} e^{-2it} \sum_{\substack{\text{all pairs}\\ \text{such that}\\ s > t}} \rho_{st}\ \sigma_s\ \sigma_t\, e^{-i(s+t)}. \tag{31}$$

If D_s for each period, s, is independent of D_t for each other period, t, then $\rho_{st} = 0$ for all s, t, and Equation (31) reduces to

$$V_D(PV(\$)) = \sum_{t=1}^{T} \sigma_t^2 e^{-2it}. \qquad (32)$$

If, on the other hand, D_s is perfectly correlated with D_t for all s, t combinations, then $\rho_{st} = 1$ for all s, t and Equation (31) reduces to:

$$VD(PV(\$)) = \sum_{t=1}^{T} \sigma_t^2 e^{-2it} + 2 \sum_{\substack{\text{all pairs} \\ \text{such that} \\ s > t}} \sigma_s \sigma_t e^{-i(s+t)} = \left[\sum_{t=1}^{T} \sigma_t e^{-it}\right]^2. \qquad (33)$$

Hillier (48) proposes a model for D which is a combination of the correlation extremes shown in Equations (32) and (33). This model makes allowance for the fact that D_t might be made up of multiple cash flows, some of which are perfectly correlated with the corresponding cash flows in the other periods. Therefore, the assumption is made that D_t consists of normally distributed random variables Y_t, $Z_t^{(1)}$, $Z_t^{(2)}$, ..., $Z_t^{(m)}$ such that

$$D_t = Y_t + Z_t^{(1)} + Z_t^{(2)} + \ldots + Z_t^{(k)} + \ldots + Z_t^{(m)}. \qquad (34)$$

The new random variables are mutually independent except that $Z_0^{(k)}$, $Z_1^{(k)}$, ..., $Z_t^{(k)}$ are perfectly correlated for k = 1, 2, ..., m. Hillier shows that the variance of the present worth of these cash flows is:

$$V_D(PV(\$)) = \sum_{t=0}^{T} [\frac{V(Y_t)}{(i+i)^{2t}}] + \sum_{k=1}^{m} \left(\sum_{t=0}^{T} [\frac{\sqrt{V(Z_t^{(k)})}}{(1+i)^t}] \right)^2 \qquad (35)$$

where $(1+i)^{-t}$ is the discrete equivalent of e^{-it}. As a side note, the expected value of the present worth of the model shown in Equation (34) is:

$$E(PV(\$)) = \sum_{t=0}^{T} [\frac{E(D_t)}{(1+i)^t}] = \sum_{t=0}^{T} \frac{E(Y_t) + \sum_{k=1}^{m} E(Z_t^{(k)})}{(1+i)^t} \qquad (36)$$

While Hillier's model has the appeal of greater potential accuracy due to a finer breakdown of D_t, it may be impractical because of the detail of the estimates required.

If the dispersion of the annual cash flow is being considered by a practitioner, there would not usually be enough information available to enable him to estimate such quantities as ρ_{st} and σ_t^2 accurately. Hence, it seems reasonable to postulate that in practice these quantities would usually be assumed to be constants throughout the life. For this reason and because of the limitless combinations of ρ_{st} and σ_t^2 which could conceivably occur, the next section will concentrate on cases where σ_t^2 and ρ_{st} are treated as constants over the entire project life.

Comparison of $V_D(PV(\$))$ and σ_t^2. Figures 43-44 are given to illustrate the effect of various coefficients of correlation, ρ_{st}, on the ratio $V_D(PV(\$))/\sigma_t^2$, which is denoted as W_ρ. In this ratio, σ_t^2 and ρ_{st} are assumed to be constant throughout the series. W_ρ is shown for a continuous range of lives from 2 to 30 years and for interest rates of 5 per

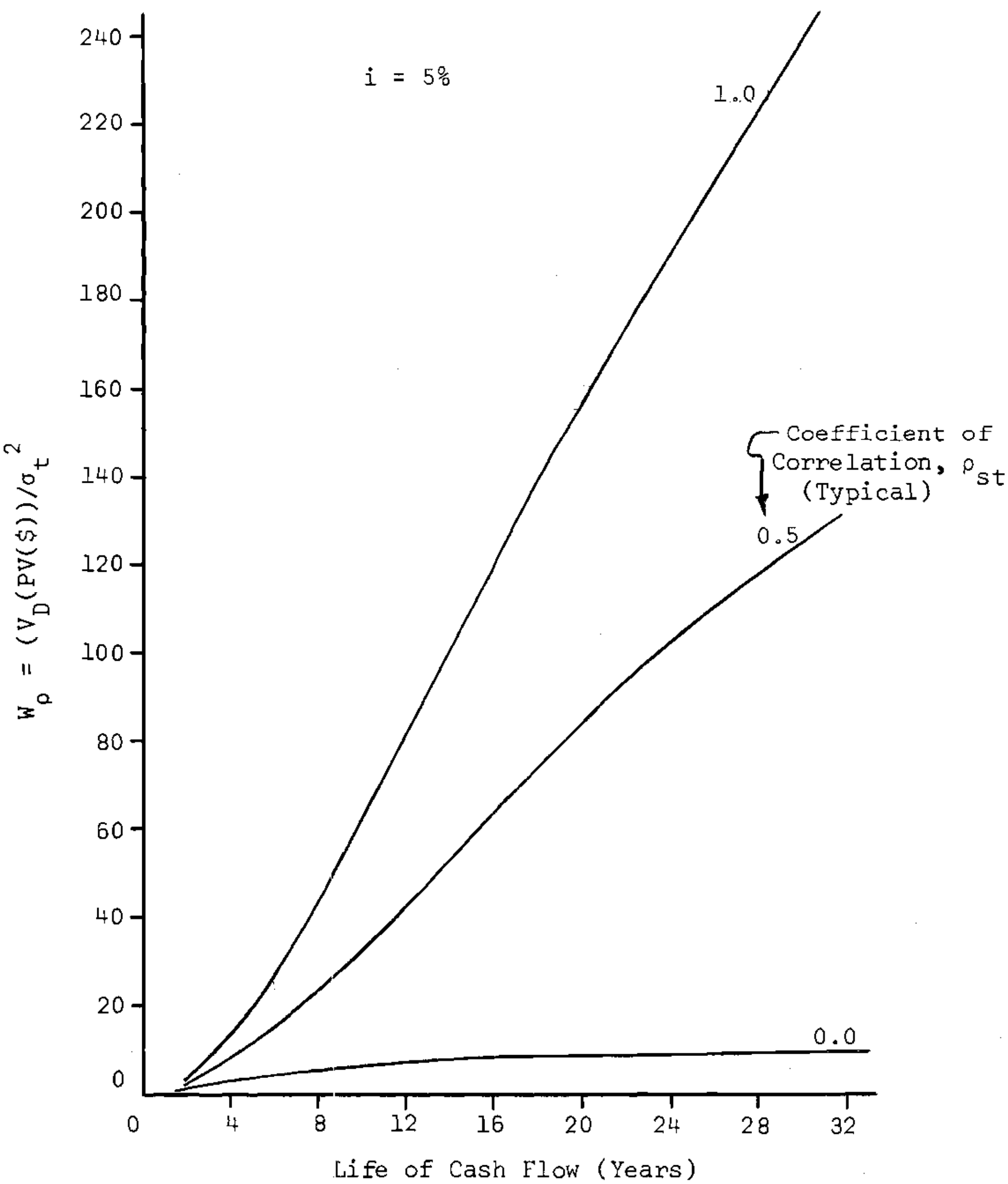


Figure 43. Variance of Present Value Compared to Variance of Annual Cash Flow for a Uniform Series of Cash Flows

(Constant σ_t^2 and ρ_{st} Throughout Series)

$$W_\rho = (V_D PV(\$))/\sigma_t^2$$

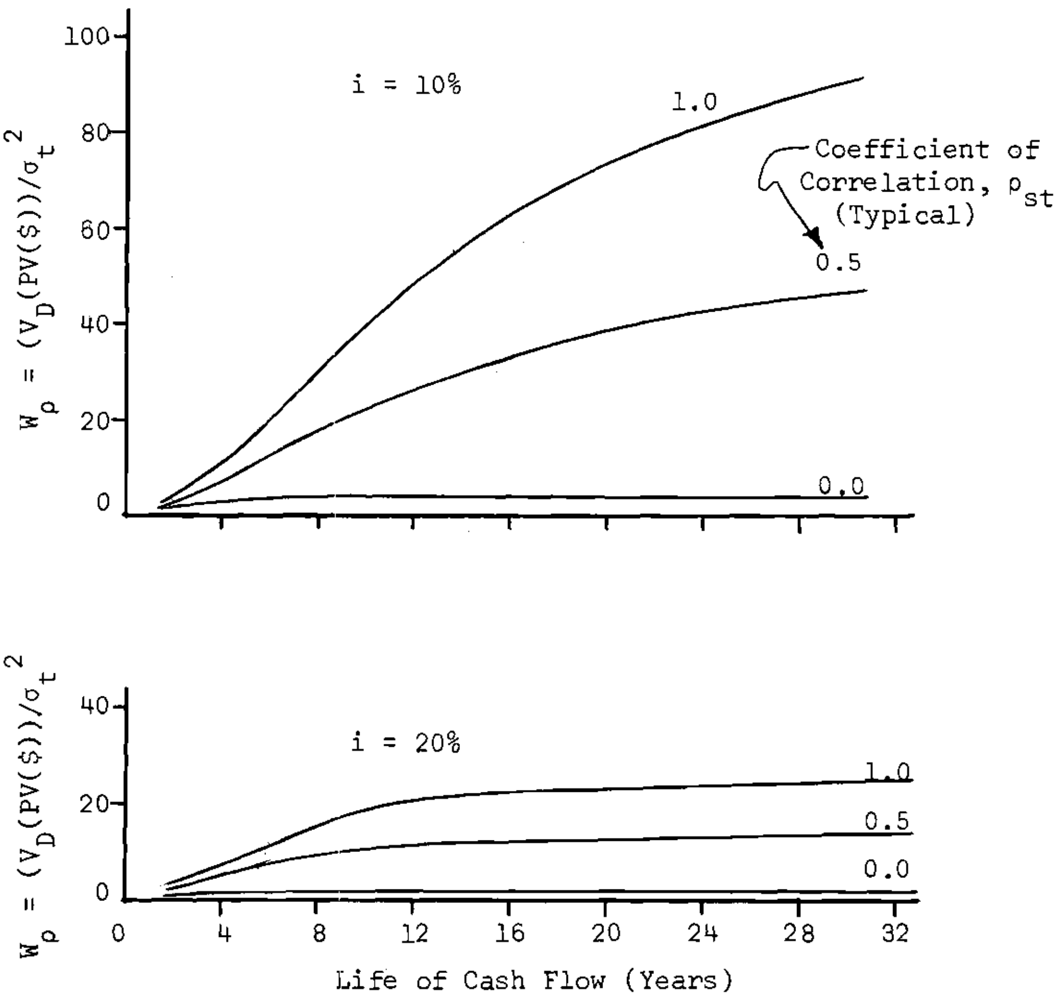


Figure 44. Variance of Present Value Compared to Variance of Annual Cash Flow for a Uniform Series of Cash Flows

(Constant σ_t^2 and ρ_{st} Throughout Series)

$W_\rho = (V_D PV(\$))/\sigma_t^2$

cent, 10 per cent, and 20 per cent. Note that W_ρ is greatest for $\rho_{st} = 1.0$ and least for $\rho_{st} = 0.0$. The difference between W_ρ for $\rho_{st} = 1.0$ and for $\rho_{st} = 0.0$ is contained in the right hand term of Equation (31), which is a linear function of ρ_{st}. Thus, W_ρ can be found by the relation:

$$W_\rho = W_{\rho=0} + \rho_{st}[W_{\rho=1} - W_{\rho=0}] \tag{37}$$

Figures 43 and 44 are useful for relating the magnitude of the $V_D(PV(\$))$ for a uniform annual series of cash flows to the magnitude of the variance of the cash flow for each year over a wide range of lives, interest rates, and coefficients of correlation.

Distribution of PV($)

The methods for approximating the mean and variance of PV($) shown in Equations (23) and (26) assume mutual independence of the individual elements. However, they do not depend upon the individual elements assuming any particular distribution. That is, if the mean and variance but not the distribution of each of the individual elements is known, the methods shown allow one to calculate an approximation of the mean and variance of PV($), even though the shape of the distribution of PV($) would not be known.

One useful characteristic of the shape of the distribution of PV($) is if each of the individual elements are independent and normally distributed, then the combination of those elements into PV($) is approximately normally distributed. It is doubtful if each of the elements would actually prove to be normally distributed in the long run. However, it seems reasonable that very often the best subjective probability distribution that can be estimated for an element would be an approximately symmetrical distribution resembling the normal distribution.

Even if the shape of the distribution of PV($) cannot be determined with any confidence, the mean and variance of PV($) alone provide substantial information for evaluating and comparing project(s). Weak probability statements can be made using inequality theorems such as those attributed to Tchebycheff and Camp-Meidel. The only consequence of not knowing the shape of PV($) is that precise probability statements cannot be made.

Example to Demonstrate Formulas for Calculating E(PV($)) and V(PV($))

Given. $\hat{E}(P) = -\$100{,}000$, $\hat{V}(P) = (\$1{,}000)^2$;

$\hat{E}(T) = 10$ years, $\hat{V}(T) = (1 \text{ year})^2$;

$\hat{E}(S) = +\ \$10{,}000$, $\hat{V}(S) = (\$2{,}500)^2$;

$\hat{E}(D) = +\ \$15{,}700$, $\hat{V}(D) = (\$2{,}000)^2$;

interest is 10 per cent and each of the above elements is independent of the other.

Solution.

$$E(PV(\$)) = -\$100{,}000 + \$15{,}700\ (6.32(.99)) + \$10{,}000\ (.360) = +\$1{,}830$$

$$V(PV(\$)) = \left(\frac{\partial PV(\$)}{\partial P}\right)^2 \cdot V(P) + \left(\frac{\partial PV(\$)}{\partial D}\right)^2 \cdot V(D) + \left(\frac{\partial PV(\$)}{\partial T}\right)^2 \cdot V(T) + \left(\frac{PV(\$)}{\partial S}\right)^2 \cdot V(S);$$

$(\partial PV(\$)/\partial P)^2 = (1)^2 = 1$,

$(\partial PV(\$)/\partial D)^2 = 39$ (From Figure 41),

$(\partial PV(\$)/\partial T)^2 = (C_1 - C_2)^2 = (.37(\$15{,}700) - .37(\$10{,}000)(.10))^2$

$= (\$5{,}439)^2$ (From Figure 39),

$(\partial PV(\$)/\partial S)^2$ = .15 (From Figure 40);

$$V(PV(\$)) = 1(\$1{,}000)^2+39(\$2{,}000)^2+(\$5{,}439)^2(1)^2+.15(\$2{,}500)^2$$
$$= 187{,}520{,}000.$$

Standard deviation of PV($) = $\sqrt{187{,}520{,}000}$ = $13,700.

From these results, it can be surmised that the project is barely acceptable based on E(PV($)) = $1,830 > 0. However, the relatively high variance indicates considerable dispersion which, together with the low E(PV($)), means that there is a relatively high probability of the project turning out to be unacceptable. If, for example, each of the individual elements are estimated to be normally distributed so that PV($) is approximately normally distributed, then the calculation of the approximate probability that the project will turn out to be unacceptable (i.e., PV($) < 0) is:

$$P(PV(\$) < 0) = P[k' < \frac{0-1{,}830}{13{,}700}] = 0.45 \qquad (38)$$

where k' is the standard normal deviate. Thus, if the distribution of PV($) is normal and other conditions are as stated in the problem, there is 45 per cent chance that the project will turn out to have a negative PV($), and thus be unacceptable.

Consideration of Correlation Between Elements

All the work in this chapter has assumed mutual independence between each of the elements affecting the PV($) for a project. Dependence or correlation among the elements can be taken into account if that is thought to be worthwhile. Below is a discussion of how covariance between elements can be taken into account in the calculation of E(PV($)) and V(PV($)).

Effect of Correlation on E(PV($))

The effect of covariance as well as variance of the individual elements on E(PV($)) can be determined exactly through evaluation of the expected value of joint density functions. However, the estimation and specification of these joint density functions would normally be quite difficult for the practitioner to perform. It seems more feasible to use the Taylor Series expansion and to evaluate higher order terms of the expansion so as to approximate the effect of lack of independence.

Effect of Correlation on V(PV($))

Equation (26) shows a means of approximating V(PV($)) when the elements are considered to be mutually independent. If correlation exists between any pair of elements in the analysis, say T and S, the effect of that correlation on V(PV($)) can be approximated by adding a term of the form

$$2\left(\frac{\partial PV(\$)}{\partial T}\right)\cdot\left(\frac{\partial PV(\$)}{\partial S}\right)\cdot \mathrm{Cov}(T,S)$$

to Equation (26) for each pair of elements which are not independent. Cov(T,S) denotes the covariance between elements T and S. To aid in performing computations, the squares of the partial derivatives with respect to T, S, and D are shown in Figures 29-31. The covariance between any two elements, say again T and S, can be estimated by using the relation

$$\mathrm{Cov}(T,S) = \rho_{T,S}\sqrt{V(T)}\,\sqrt{V(S)} \qquad (39)$$

where $\rho_{T,S}$ is the coefficient of correlation between T and S. Suggestions to facilitate the estimation of ρ are shown in Chapter VIII.

If it were desired to consider lack of independence between all possible pairs of the four elements in Equation (26), there would be $\binom{4}{2} = 6$ terms of the form shown above. In most economic analyses most of these terms would be either insignificant in amount or difficult to estimate with reasonable confidence, and hence would be neglected. The term most likely to be significant is the term which considers the correlation of D and T. Intuitively, it seems that a larger D (higher net receipts or lower net disbursements) may well lead to a longer life than the unconditional expected life, while a smaller D may lead to a shorter life than the unconditional expected life. If this is true, then there would be a relatively high positive $\rho_{D,T}$. Further, $\partial PV(\$)/\partial D$ and $\partial PV(\$)/\partial T$ would be relatively high, which together with a high $\rho_{D,T}$ would result in a high value for the term which considers the correlation of D and T.

Summary

This chapter has covered means of considering the effect of variation of multiple elements in economic analyses. The developments herein apply to analyses for single projects. The next chapter shows how to use the results of this chapter in the comparison of projects.

CHAPTER V

PROCEDURES FOR COMPARING PROJECTS WHEN CONSIDERING VARIATION OF MULTIPLE ELEMENTS

Introduction

Computational means for considering the effect of dispersion of multiple elements in economic analyses of individual projects by the present value method were discussed in Chapter IV. This chapter will use those results and show how alternative projects can be compared so as to facilitate final selection. Consideration will be made of the covariance between pairs of projects when comparing mutually exclusive projects.

The present value method for comparing alternatives requires a common study period for all projects being compared. If the lives of the projects being compared differ, the present value method can be used to consider what happens to each project over a study period equal to a common multiple of the lives of the projects or the length of needed service, whichever is less. The last chapter was devoted to computational procedures for finding PV($) for just one life cycle of a project. However, if the lives differ, E(PV($)) and V(PV($)) can be determined by the same procedures over the entire length of the study period rather than just one life cycle for each project. Regardless of the length of the study period used, the selection process as outlined in the foregoing procedure steps can be used.

If the length of the study period comprising a multiple of the lives is so long that it is burdensome to consider happenings for each project over that period, it may be advantageous to use the annual worth as a measure of merit. The annual worth of a project will be denoted as AW($). If like-to-like replacements for each alternative project over the entire study period are assumed, then the AW($) calculated over one life cycle for each project is exactly proportional to the PV($) calculated over the entire study period for each project. The annual worth for one life cycle of a project may be calculated in general as:

$$AW(\$) = \sum_{t=0}^{T} X_t \, e^{-it} \cdot (i/1-e^{-iT}) \tag{40}$$

where X_t is the cash flow for the t^{th} year. If X_t is broken down into major elements and continuous compounding, continuous payments is assumed as was done to develop Equation (21) for PV($), then

$$AW(\$) = (P+S)\left(\frac{i}{1-e^{-iT}}\right) - S(i) + D. \tag{41}$$

If the distribution of PV($) for a project over a life cycle or for a given study period of n years is known, then the distribution of the equivalent annual worth for that project can be determined directly by the relation

$$AW(\$) = PV(\$) \cdot \left(\frac{i}{1-e^{-in}}\right) \tag{42}$$

where n is the number of years of the life cycle or the study period over which the PV($) was determined. If AW($) rather than PV($) is used as the measure of merit, the procedure for selection among mutually exclusive alternatives that follows is the same except that PV($) in the description is everywhere changed to AW($).

Criterion for Selection to Use with the PV($) Method

In assumed certainty studies using the PV($) method, the decision rules for selection that are most commonly advocated are based on the amount of net PV($). However, in cases where revenue or savings as well as disbursements are known, there are grounds for argument that the decision rules should be based on the ratio of net present value to investment. Barish (4-p.234), who uses premium worth to mean net positive present value and premium worth percentage to mean the ratio of net positive present value to the investment, says: "Either one can be used as a capital rationing criterion: all proposals (projects) with a positive premium worth or premium worth percentage would be accepted; when proposals are mutually exclusive (alternative), those with the highest premium worth or premium worth percentage would be accepted."

In the procedures for selection which will be covered herein, the decision rules are based on the amount of net PV($), although in cases where revenues or savings are known, they could be based on the ratio of net PV($) to investment. Similarly, if the AW($) were being used and revenue or savings were known, the decision rules could be based on the ratio of net positive AW($) to investment rather than just the amount of AW($).

In the section which follows, a procedure for selecting one of a group of mutually exclusive projects will be shown. After that, a procedure for selecting from a group of non-mutually exclusive projects will be presented. Examples of the use of each of these procedures will be demonstrated.

A Procedure for Selection Among Mutually Exclusive Projects

Procedure Step 1

Calculate estimated mean and variance of PV($) for each project considered by using the methods shown in Chapter IV. Call the mean and variance of PV($) for project x $E(PV(\$))_x$ and $V(PV(\$))_x$, respectively.

Procedure Step 2

For the projects with the most positive E(PV($)) and second most positive E(PV($)), calculate the expected difference, $E(PV(\$))_d$, and the variance of the difference, $V(PV(\$))_d$. The expected difference between projects x and y may be calculated as

$$E(PV(\$))_d = E(PV(\$))_x - E(PV(\$))_y. \tag{43}$$

If the PV($) of the two projects are mutually independent, then the variance of the difference can be calculated as

$$V(PV(\$))_d = V(PV(\$))_x + V(PV(\$))_y. \tag{44}$$

If there is dependence regarding the cash flows of the individual projects, the variance of the difference between projects x and y may be calculated as

$$V(PV(\$))_d = V(PV(\$))_x + V(PV(\$))_y - 2\ Cov(PV(\$)_x, PV(\$)_y). \qquad (45)$$

The covariance of the PV($) of cash flows for projects x and y might be estimated most effectively through estimation of the coefficient of correlation, ρ_{xy}.[6] If this is done, then the variance of the difference can be calculated as

$$V(PV(\$))_d = V(PV(\$))_x + V(PV(\$))_y - 2\rho_{xy}\sqrt{V(PV(\$))_x}\ \sqrt{V(PV(\$))_y}\ . \qquad (46)$$

Figure 45 serves to demonstrate the relative effect on $V(PV(\$))_d$ caused by different coefficients of correlation and different relative proportions of $\sqrt{V(PV(\$))_x}$ and $\sqrt{V(PV(\$))_y}$. The primary quantity which can be obtained from this figure is the ratio of the variance of difference between two distributions under consideration to the variance of difference between two independent, equal-variance distributions having the same sum of standard deviations as the two distributions of interest. Note in Figure 45 how the ratio decreases as ρ_{xy} increases to the limit of +1.0, and increases to a maximum of 2.00 as ρ_{xy} decreases to the limit of -1.0. Note also how, for a given ρ_{xy}, the ratio increases as the standard deviation of each distribution expressed as a per cent of the sum of the standard deviations for both distributions differs increasingly from 50 per cent - 50 per cent relative proportions. As a limit, the

6. See Chapter VIII for comments on the estimation of ρ.

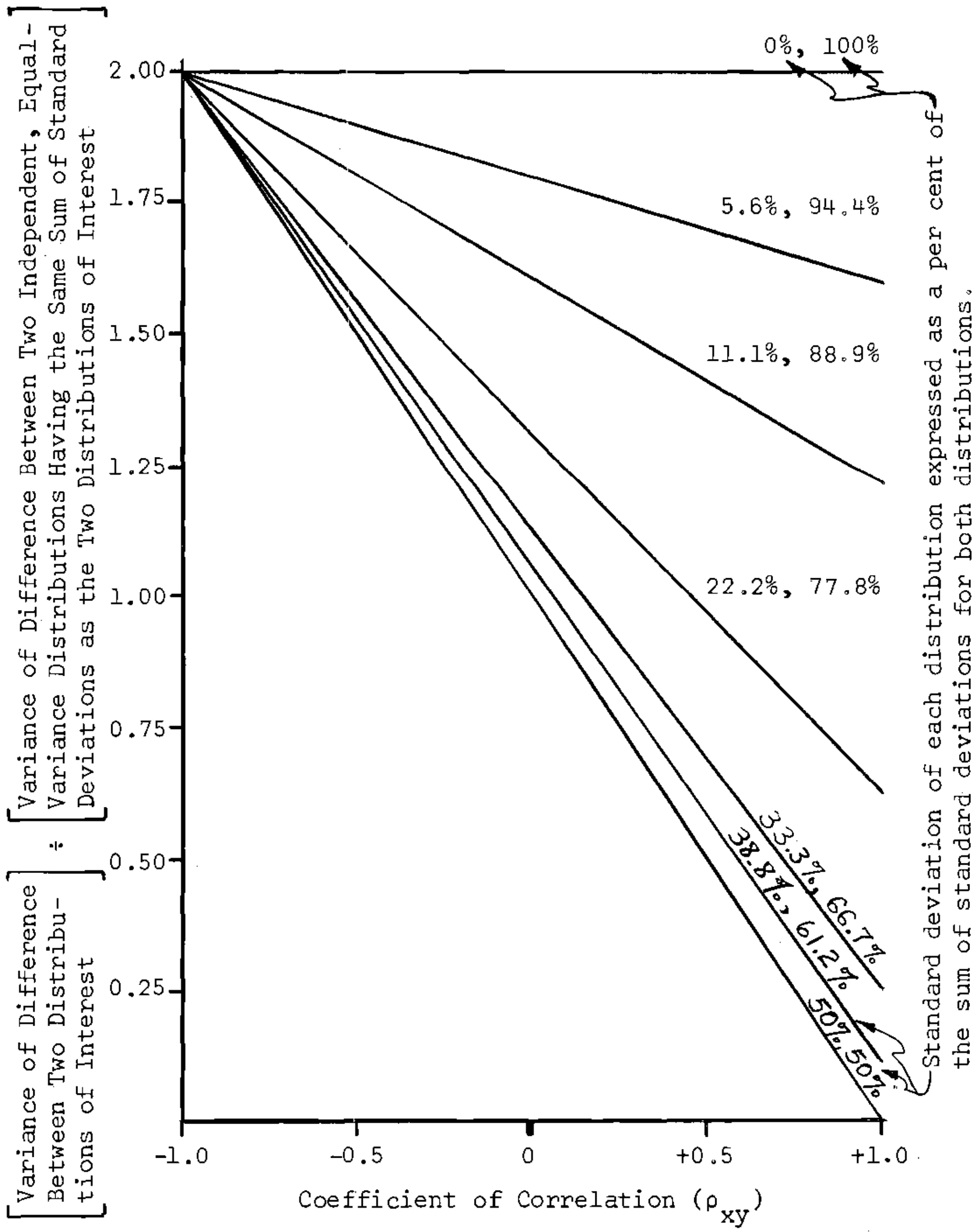


Figure 45. Comparison of Variance of Difference Between Two Distributions for Various ρ_{xy} and Relative Proportions of Standard Deviations for the Two Distributions

ratio becomes 2.00 for all ρ_{xy} when the standard deviations of the two distributions assume 0 per cent - 100 per cent relative proportions.

Figure 46 is provided to demonstrate the behavior of $V(PV(\$))_d$ for $\rho_{xy} = 1.0$ over a continuous range of $V(PV(\$))_x$ and $V(PV(\$))_y$. Figure 47 considers the effect of the correlation between x and y, and can be used to graphically aid in determining $V(PV(\$))_d$ for any value of ρ_{xy}. Note from the explanation in Figure 47 that:

$$V(PV(\$))_d = V(PV(\$))_x + V(PV(\$))_y + \rho_{xy} \text{ (Value Found in Figure 47)}. \quad (47)$$

These figures should be useful to the practitioner who desires to obtain fast approximations of $V(PV(\$))_d$.

Procedure Step 3

With the information obtained in Procedure Step 2 (above) determine the "probability of reversal." Probability of reversal means herein the probability that the project with the more positive E(PV($)) will actually turn out to be not as good as the project with the less positive E(PV($)). Figure 48 shows how this probability of reversal is generated. The distributions of PV($) outcomes for two projects, x and y, are shown on the top of the figure. Though project x has a higher E(PV($)) than does project y, there is a positive probability that a random observation of PV($) is less than a random observation of $PV(\$)_y$. The distribution of $PV(\$)_d = PV(\$)_x - PV(\$)_y$ is shown in the bottom of the figure. The probability that $PV(\$)_d < 0$, the probability of reversal, is shown by the shaded area.

<u>To Use:</u> Find coordinate point corresponding to the variances of the two distributions. Visually interpolate to find the variance of the difference.

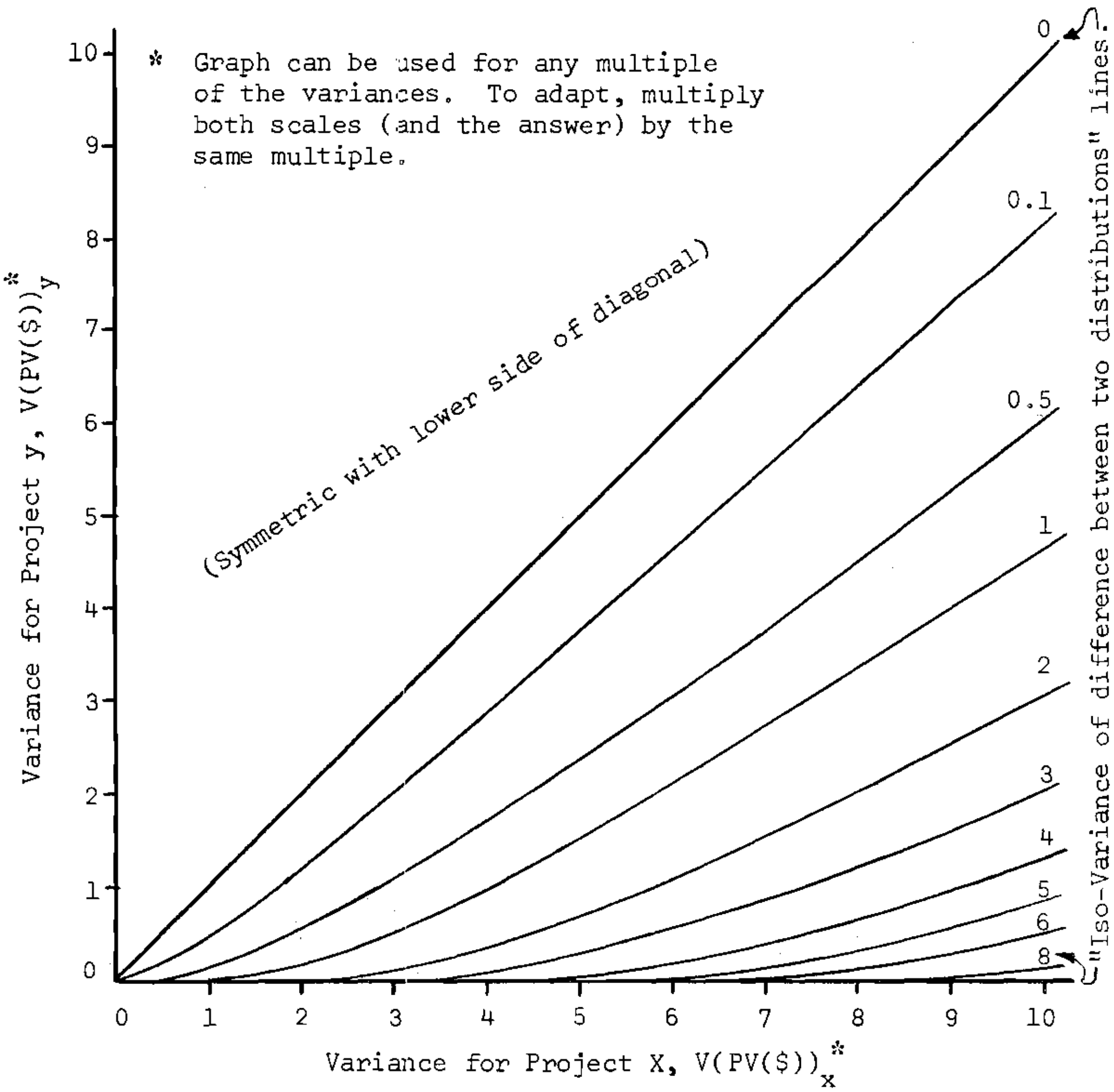


Figure 46. Variance of Difference Between Two Distributions Which are Perfectly Correlated (ρ = 1)*

* To Use: (1) For given variance for x and y, find maximum effect of correlation from graph below.

(2) Multiply maximum effect of correlation by coefficient of correlation, ρ, to find effect of correlation = $-2\rho\sqrt{V(PV(\$))_x}\ \sqrt{V(PV(\$))_y}$.

** To Find Total Variance of Difference Between Two Distributions: Add effect of correlation (from above) to the sum of the variances of the two distributions.

*** Graph can be used for any multiple of variances. To adapt, multiply both scales (and the answer) by the same multiple.

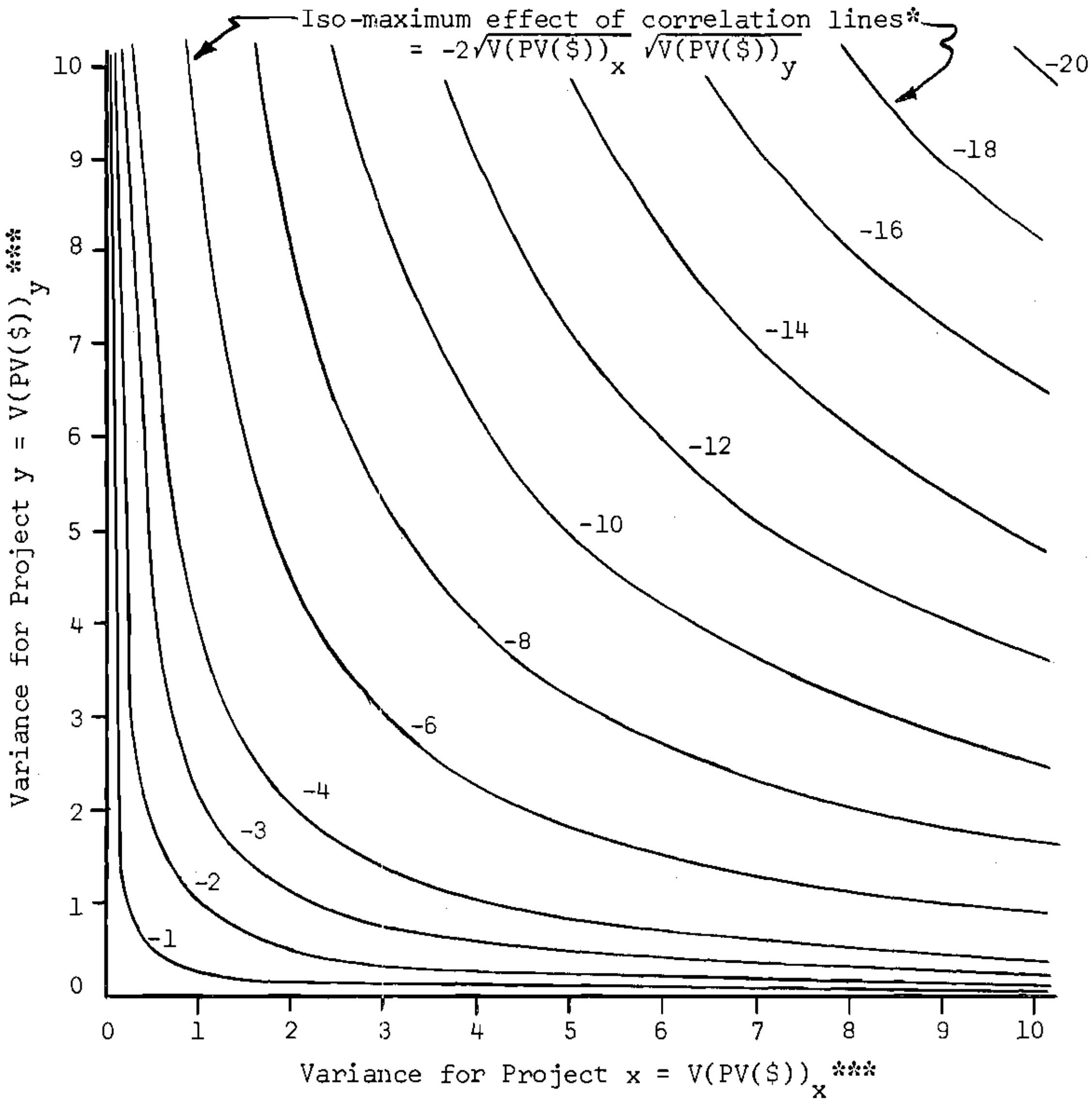
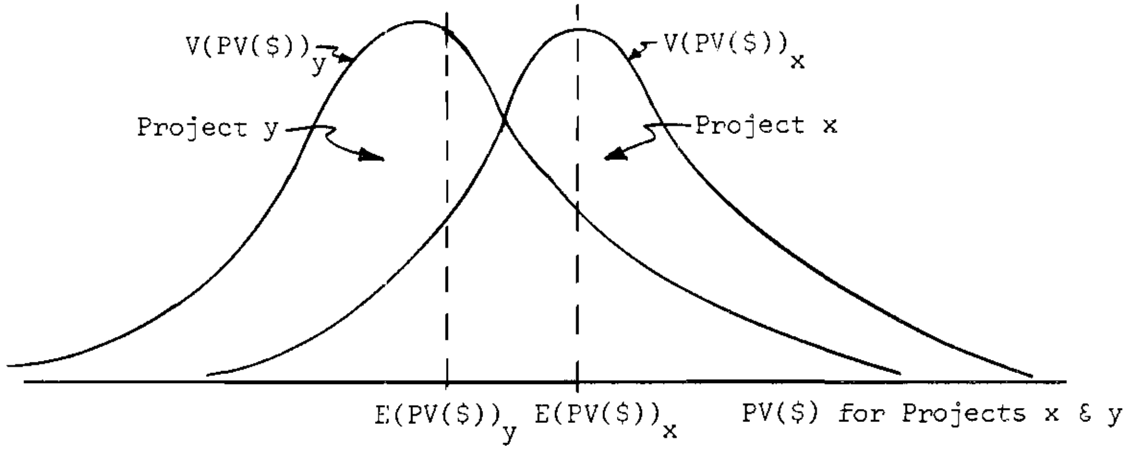
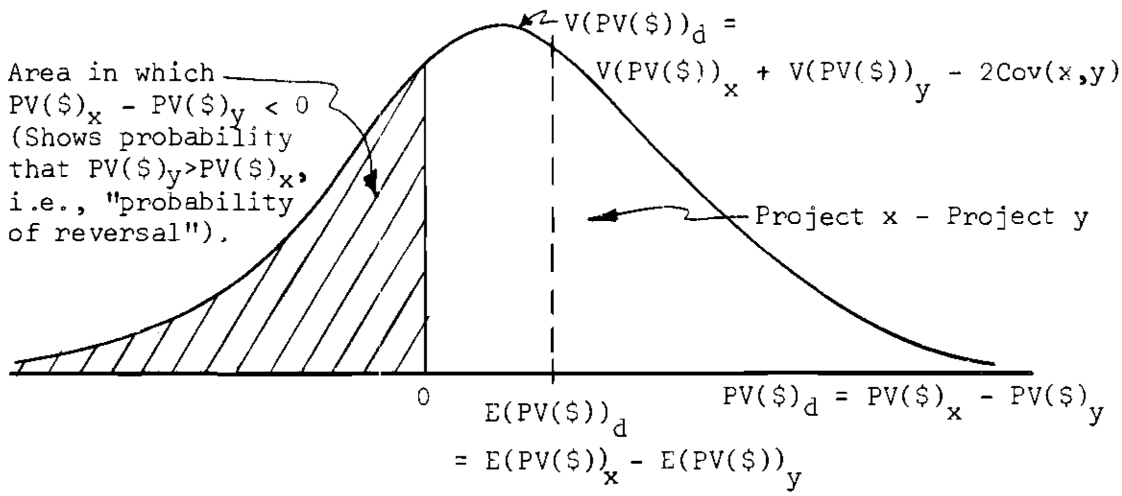


Figure 47. Effect of Correlation* on Variance of Difference Between Two Distributions**



Distributions of PV($) for Two Projects



Distribution of PV($) for Difference Between Two Projects

Figure 48. Demonstration of Meaning of Probability of Reversal

To finish procedure step 3, the decision-maker should weigh the probability of reversal together with information on $E(PV(\$))_d$ and judgment on irreducible or intangible factors to decide which of the two projects is the more desirable.

Shape of Distribution. While $E(PV(\$))_d$ and $V(PV(\$))_d$ can be calculated by Equations (43) and (45), respectively, without regard to the shape of distribution for the individual projects, it is necessary to know also the density function of $PV(\$)_d$ in order to be able to calculate the probability of reversal. The density of $PV(\$)_d$ depends upon the densities of PV($) for projects x and y as well as ρ_{xy}.

If $PV(\$)_x$ and $PV(\$)_y$ are normally distributed and independent of each other ($\rho_{xy} = 0$), then $PV(\$)_d$ is also normally distributed. If $PV(\$)_d$ is normally distributed, then it is easy to determine probabilities of reversal by using the relation:

$$P(\text{reversal}) = P[k' \leq \frac{0-E(PV(\$))_d}{\sqrt{V(PV(\$))_d}}] \tag{48}$$

where k' is the standard normal (N(0,1)) variable, which is commonly tabled.

Figure 49 is given to enable one to graphically determine the probability of reversal for a normal distribution of differences with a given mean and standard deviation. It is also useful in aiding one to visualize the effect of different means and standard deviations on the probability of reversal.

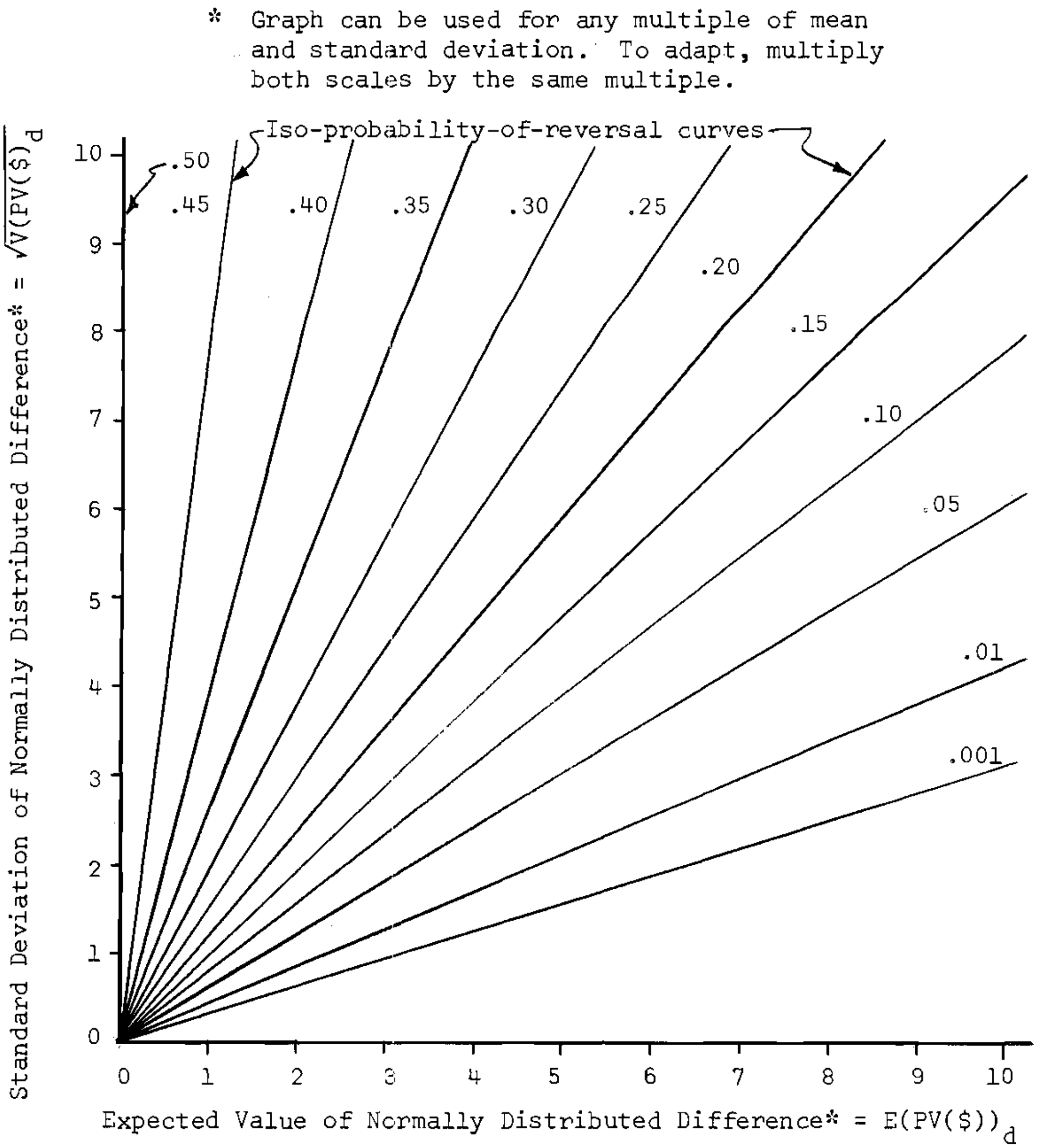


Figure 49. Probability of Reversal According to Mean and Standard Deviation of Normal Distribution of Difference Between Two Projects

Figures 50 and 51 give two aids for studying the effects on the probability of reversal caused by errors in the estimated standard deviation of the distribution of $PV(\$)_d$ when that distribution is normal. Figure 50 can be used to visually examine this effect when the actual standard deviation differs from the estimated standard deviation of the distribution by certain percentages up to ± 50 per cent. Figure 51 shows similar information, except that the results are expressed as ratios of probabilities of reversal. These figures can be quite useful for aiding the analyst in performing sensitivity studies concerning the probability of reversal.

Procedure Step 4

Once the more preferable of the two projects having the most positive E(PV($)) has been selected as outlined in procedure step 3, compare this project against the project having the next less positive E(PV($)). Make the comparison by repeating procedure step 3. This will result in a probability of reversal for this new pair of projects that can be compared with the $E(PV(\$))_d$ and judgment on intangibles to decide which of these two projects is more desirable.

Procedure Step 5

The procedure outlined in steps 2 through 4 can be repeated as long as alternative projects exist and it seems that the project having the next less positive E(PV($)) is sufficiently competitive to the last most desirable project to warrant consideration of probability of reversal and judgment on intangibles. The final choice should be the more desirable project remaining after the succession of comparisons. In other words, the final choice should be the project which has the most

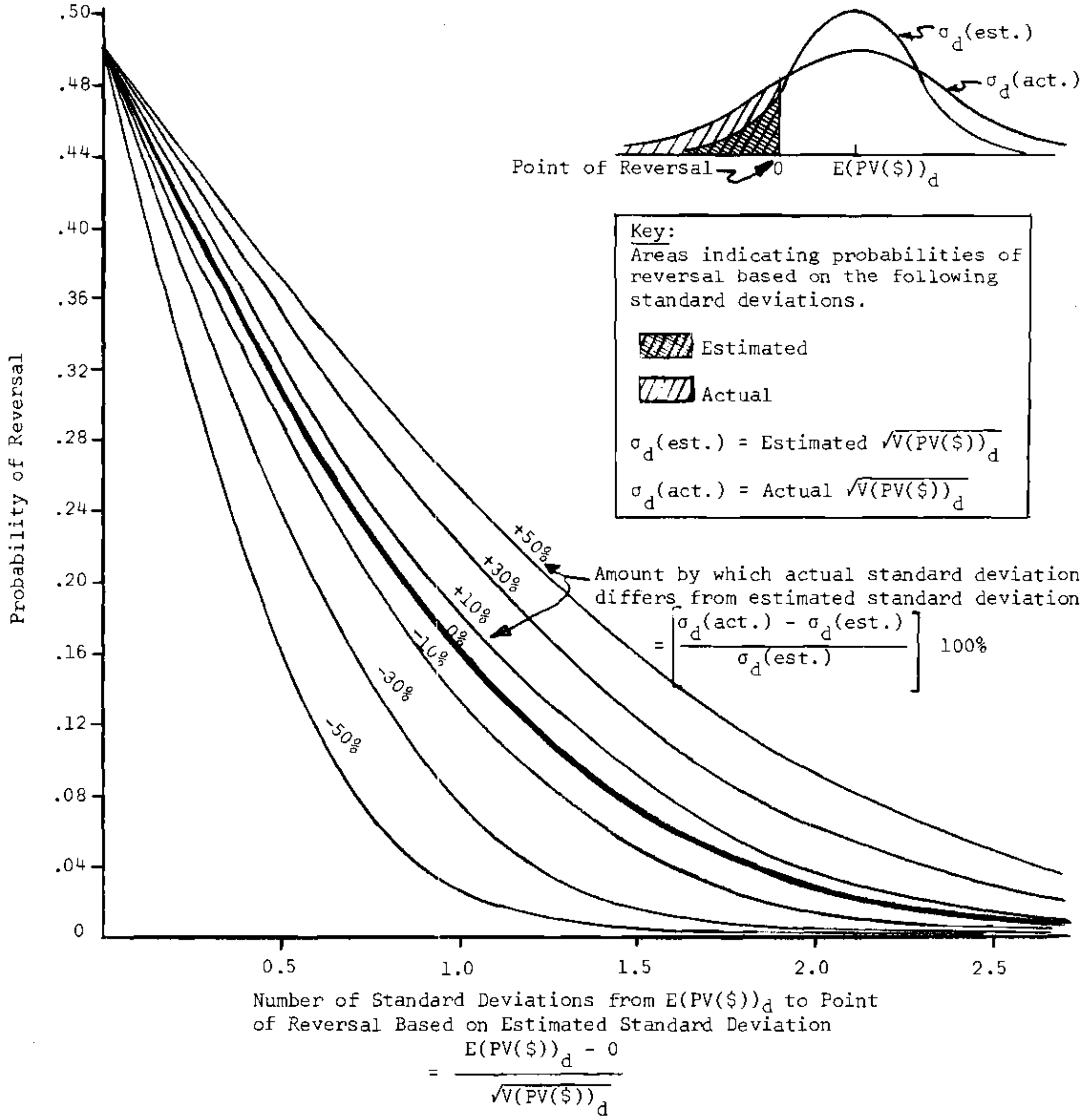


Figure 50. Change in Probability of Reversal Caused by Error in Estimate of Standard Deviation for Normal Distribution of $PV(\$)_d$

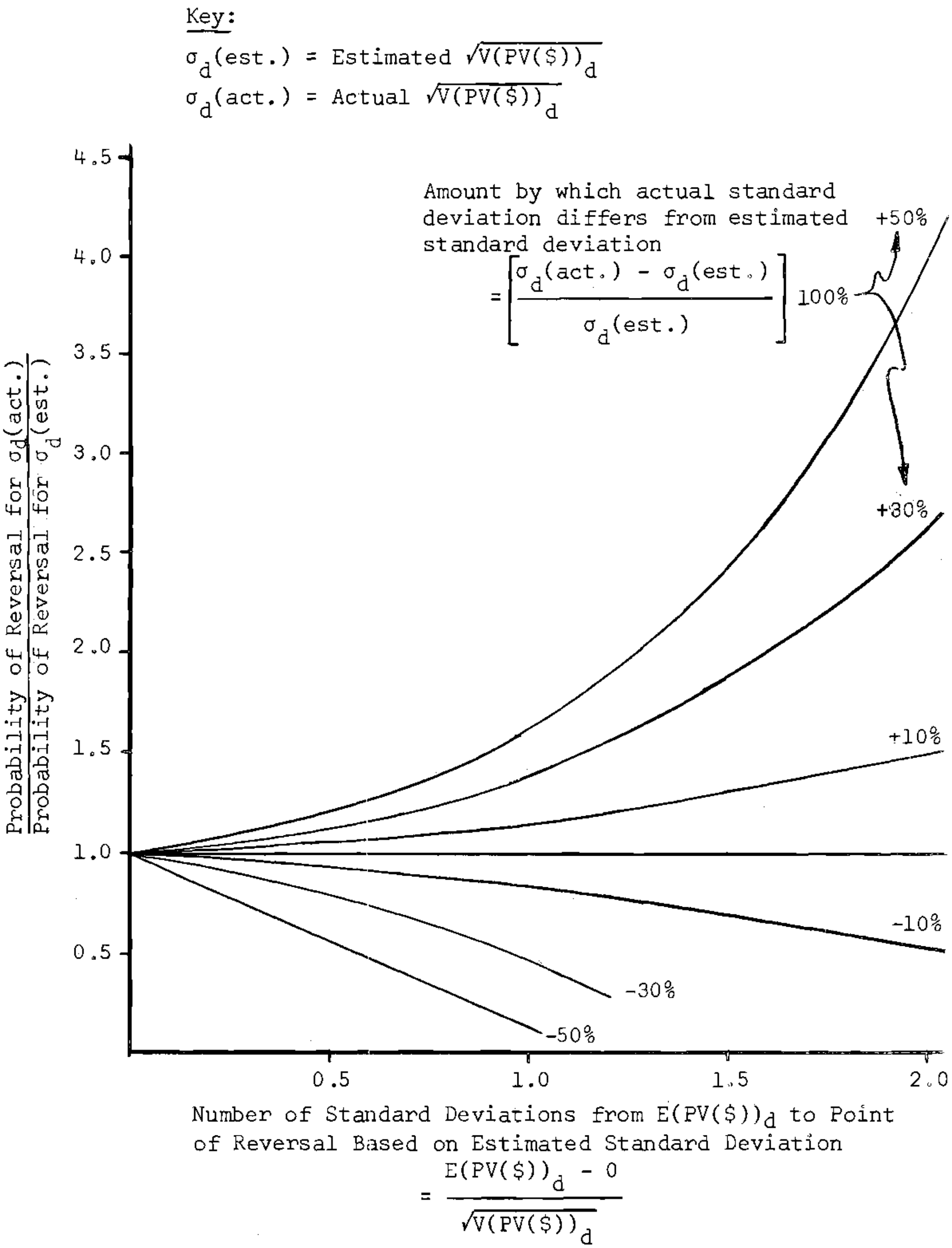


Figure 51. Relative Error in Probability of Reversal Caused by Error in Estimate of Standard Deviation for Normal Distribution of $PV(\$)_d$

positive E(PV($)) <u>and</u> which has not been judged to be less desirable than some other project due to consideration of probability of reversal and intangibles.

Example of Use of Procedure for Selection Among Mutually Exclusive Projects

Procedure step 1 calls for estimation or calculation of the mean and variance of PV($) for each project. These results to be used in this example are shown in Table 6. Assume that the PV($) for each project is normally distributed and independent. Hence, the distribution of differences between any two of the projects is normal.

Table 6. Data on Mutually Exclusive Projects

Project	$\hat{E}$(PV($))	$\hat{V}$(PV($))
A	$+2,000	144,000,000
B	+600	187,540,000
C	+100	36,000,000
D	+50	64,000,000

Procedure step 2 calls for comparison of Projects A and B. This would consist of:

$$E(PV(\$))_d = \$2{,}000 - \$600 = \$1{,}400,$$

$$V(PV(\$))_d = 144{,}000{,}000 + 187{,}540{,}000 - 0 = 331{,}540{,}000, \text{ and}$$

$$\sqrt{V(PV(\$))_d} = \$18{,}200.$$

Procedure step 3 calls for calculation of the probability of reversal. Since the distribution of differences is normal, Figure 49 can be used to find that the probability of reversal = 0.47. This quite high probability of reversal together with consideration of the relative effects of intangibles on projects A and B should now be weighed by the decision-maker and a choice between the two projects should be made. For purposes of this illustration, suppose that it is decided that intangibles look very slightly more favorable to project B than to project A, but that this consideration together with the 47 per cent probability of reversal is not enough to swing the choice to B. Hence, of the two, A is selected as the more desirable.

Procedure step 4 involves the comparison of the more desirable project selected in the last step, project A, with the project with the next less positive E(PV($)), project C. This is done below:

$$E(PV(\$))_d = \$2{,}000 - 100 = \$1{,}900,$$

$$V(PV(\$))_d = 144{,}000{,}000 + 36{,}000{,}000 = 180{,}000{,}000, \text{ and}$$

$$\sqrt{V(PV(\$))_d} = \$13{,}400.$$

Using Figure 49, the probability of reversal = 0.44. This should now be considered together with the relative effects of intangibles to decide which is the more desirable project. Suppose that intangibles look markedly more favorable to project C than to project A so that the consideration of the probability of reversal together with these intangibles results in project C being selected as the more desirable.

Procedure step 5 involves successive comparisons of the more desirable project selected in the last step with the project with the next less positive E(PV($)). In this example problem, there is only one more project with a less positive E(PV($)) to consider, project D. Project C compared to project D results in an $E(PV(\$))_d$ of $50 and a $V(PV(\$))_d$ of 100,000,000. From Figure 49, these conditions result in a probability of reversal of almost 0.50. If intangibles are, say, more favorable to project C, then project C would be the more desirable of the two projects. Concluding procedure step 5, the final choice among the alternative projects is project C.

It should be noted that this procedure is just as applicable to selection among projects where costs only are known. In such cases, the project with the most positive PV($) is actually the project with the least PV($) of costs, etc.

A Procedure for Selection Among Non-Mutually Exclusive Projects

In selecting among non-mutually exclusive projects, the only projects on which the economics can be evaluated are those for which revenues or savings as well as costs can be estimated. If costs only are known, then there is no way to quantitatively say that a project has a positive net PV($) and is thus earning at least the minimum required rate of return.

Procedure Step 1

Calculate estimated mean and variance of PV($) for each project considered by using the methods shown in Chapter IV.

Procedure Step 2

Rank the projects in order of decreasing net E(PV($)). If investment funds are limited, include information on investment requirements.

Procedure Step 3

For each project, determine the "probability of loss" (i.e., the probability that PV($) will turn out to be < 0). This probability is analogous to, and can be calculated in the same manner as, the probability of reversal for mutually exclusive projects shown in the last section. The only difference is that in this case the distribution with which to work is the distribution of PV($) for an individual project rather than the distribution of $PV(\$)_d$ for pairs of projects.

If the distribution of PV($) for a project is normal, the probability of loss can be determined easily by use of Figure 49, and Figures 50 and 51 can be used in examining the relative effects of errors in the estimated standard deviation of PV($). In using Figures 49-51 for an individual project, "reversal" is everywhere changed to "loss" and parameters of the distribution of differences between two projects (like $E(PV(\$))_d$ and $V(PV(\$))_d$) are everywhere changed to the corresponding parameters of the distribution for the individual project.

Procedure Step 4

Start with the project at the top of the list and subjectively weigh the combined criteria of E(PV($)), probability of loss, and consideration of intangibles and make a decision on whether or not to tentatively accept that project. Continue in this manner down the list, tentatively accepting those projects which meet the combined criteria of high enough E(PV($)) and sufficiently low risk of loss together with satisfactory intangibles.

If there is no limitation on the investment funds, all projects meeting the above combined criteria can be accepted. If there is some limitation on the investment funds, then projects can be considered as above down to the point where the available funds are exhausted. After that point, projects with lower E(PV($)) values should be considered based on the same combined criteria and in light of whether or not those projects are better overall than one or more of the projects which have been tentatively accepted. If a project or projects with lower E(PV($)) values show up better, then they should be tentatively accepted and assigned the investment funds previously allocated to a project or projects with higher E(PV($)) values. This sequence is continued down the list until there are no other projects competitive enough to receive serious consideration.

The procedure for considering projects after all funds have been tentatively allocated is complicated by the fact that various projects typically require different investment amounts. Thus, it may be necessary to compare combinations of one or more projects against one or more tentatively accepted projects in order to stay within investment funds constraints and still select the group of projects which best meet the combined criteria. In judging combinations of projects to take the place of projects tentatively accepted, projects which were previously not accepted should again be considered because, in combination with others, one or more of those projects may be accepted.

Example of Use of the Procedure for Selection Among Non-Mutually Exclusive Projects

Table 7 below lists the pertinent data on five projects which are ranked in order of decreasing E(PV($)). Investment funds are limited to

$200,000, hence information on investment requirements for each project are shown. Once this information is determined and tabulated, procedure steps 1 and 2 are complete.

Table 7. Data on Non-Mutually Exclusive Projects

Project	$\hat{E}$(PV($))	$\hat{V}$(PV($))	Investment Required
Q	+5,000	36,000,000	$100,000
R	+3,000	100,000,000	$100,000
S	+2,500	169,000,000	$200,000
T	+1,000	1,000,000	$100,000
U	-2,000	100,000,000	$100,000

Procedure step 3 calls for the determination of the probability of loss for each project that has any likelihood of acceptance. Assume that the PV($) for each project is normally distributed, so that the probability of loss for each project can be determined directly from Figure 49. Note that what is called probability of reversal in Figure 49 is the same as probability of loss when considering single projects. Table 8 shows the pertinent data together with probabilities of loss for the projects under construction. Procedure step 4 calls for the successive appraisal of each project according to the combined criteria. Suppose that projects Q and R are tentatively acceptable, which uses up the investment funds available. The remaining projects must now be examined by the same procedures in light of whether they are good enough to justify taking the investment funds from projects already tentatively accepted.

Table 8. Data on Non-Mutually Exclusive Projects and Probabilities of Loss

Project	$\hat{E}(PV(\$))$	Probability of Loss	Investment Required
Q	+5,000	0.20	$100,000
R	+3,000	0.38	100,000
S	+2,500	0.42	200,000
T	+1,000	0.16	100,000
U	-2,000	0.58 (=1-0.42)	100,000

Suppose that project S is judged to not be good enough, but that project T is judged to be better overall than project R. Project U, even though it has a negative E(PV($)) may still be a serious contender if the effect of intangibles is much more favorable to it than to project Q or T. In this case, suppose that project U is not considered sufficiently good through consideration of the composite criteria, and hence projects Q and T are the final selections.

Supplementary Criteria

In choosing between mutually exclusive projects or between non-mutually exclusive projects, there are refinements to supplement the probability of loss or probability of reversal information which could be quite useful. Such refinements to be discussed herein are based on Robert Schlaifer's concepts of expected opportunity loss or cost of uncertainty (78-Chapters 7 and 30). Applied to the analysis of an individual project, the expected opportunity loss is a measure of how much could be saved, on the average, if the occurrence of the loss were perfectly predictable and the investment were not made if the loss were going to occur. The expected opportunity loss, denoted EOL, is calculated by summing the product of the loss times the probability of the loss occurring over all possible losses. Stated for the general continuous case when PV($) is the measure of merit,

$$\text{EOL} = \int_{-\infty}^{0} |PV(\$)| \cdot f(PV(\$)) \cdot dPV(\$). \tag{49}$$

If the distribution of PV($) is normal, the expected opportunity loss can be determined very readily by use of a table of the unit normal loss integral such as is in Schlaifer (78-pp.706-707). To use this table, one merely needs to determine the number of standard deviations, u, from the expected outcome to the point of loss. The formula for expected opportunity loss may then be used:

$$\text{EOL} = \sqrt{V(PV(\$))} \cdot \text{Unit Normal Loss Integral at u.}^{7} \tag{50}$$

As an example of the use of this tool, consider a project with an $\hat{E}(PV(\$)) = \$4{,}000$ and a $\sqrt{\hat{V}(PV(\$))}$ of $8,000, where PV($) is normally distributed. Thus,

$$\begin{aligned} \text{EOL} &= \$8{,}000 \times \text{Unit Normal Loss Integral at } \frac{\$4{,}000}{\$8{,}000}, \\ &= \$8{,}000 \times .1978 = \underline{\underline{\$1{,}582}}. \end{aligned}$$

Another potentially useful type of information which can now be readily determined is the expected opportunity loss if a loss does occur.

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7. For a derivation of this formula, see Appendix F.

This can be calculated as:

$$(\text{EOL} \mid \text{Loss Occurs}) = \frac{\text{EOL}}{\text{P(Loss Occurs)}} . \tag{51}$$

The probability that a loss occurs is the same thing as the "probability of reversal," which can be conveniently read from Figure 49 if the distribution is normal. In the example problem above, the probability of a loss occuring = 0.3085. Thus,

$$(\text{EOL} \mid \text{Loss Occurs}) = \frac{\$1,582}{.3085} = \underline{\underline{\$4,970}} .$$

The latter type of loss information may be more useful than the former because it gives the decision-maker a feel for what is of vital concern if he is conservative about incurring losses--a measure of the expected loss if the loss should occur. In practice, one or both of these loss criteria may be valuable supplements to the combined criteria used in judging between projects.

If applied to the comparison of two projects, the expected opportunity loss is a measure of how much could be saved, on the average, if the occurrence of a reversal could be perfectly predicted and the project with the highest E(PV($)) were not chosen if the reversal were going to occur. The determination of EOL in this case is the same as shown above except that the distribution applicable is $PV(\$)_d$ rather than PV($) for a single project.

Summary

This chapter has shown procedures for making economic analyses of both mutually exclusive and non-mutually exclusive projects when the distribution of the measure of merit, PV($), for each project is known. Special figures have been presented to facilitate the application of these procedures, particularly when the distributions are normal.

CHAPTER VI

EFFECTS OF LACK OF CERTAINTY AND FORMULATION OF DECISION GUIDES

Introduction

When element outcomes are estimated in terms of parameters of subjective probability distributions, explicit recognition is being made of the fact that those outcomes are subject to variation. An economic analysis utilizing these types of estimates can yield many benefits in added analysis information, but it is weakened by the fact that the estimates are normally subject to some degree of uncertainty or lack of certainty.

This chapter consists of an examination of the effects of this lack of certainty in the estimation of elements considered and the formulation of decision guides for recognizing situations in which the effects of random variation of elements should be considered quantitatively in economic analyses. The next section considers lack of certainty concerning project life only. The final section of the chapter considers the more important problem of lack of certainty of multiple elements in economic analyses.

As a side point, this lack of certainty which exists in the estimating of individual elements is confounded by the fact that there is normally no way to check out those individual estimates in the long run. In the case of projects accepted, there is typically no way to prove out the distributions estimated because of the lack of frequency data from sufficient projects of a kind. In the case of projects subjected to analysis but not accepted, the estimated distributions cannot be proven out in the long run merely because no outcome data would exist.

Lack of Certainty of Project Life--
Effect on Expected Analysis Outcome

Chapter III presents a rather elaborate analysis of the effects of various distributions of project life on R_{cr} and R. When projects are being compared, the effect of life dispersion that is of primary concern is the effect on R_{cr}, the ratio of the E(CRF) to the CRF at the assumed certain expected life. Examination of Figures 6-17 reveals that R_{cr} can range from 1.00 to 2.00 or higher, depending on the life distribution, coefficient of variation, interest rate, etc.

Just how much difference in R_{cr} can be caused by lack of certainty as to what is the distribution of life depends upon the projects considered in the analysis. In order for that lack of certainty to be important at all, the capital recovery cost must be a sufficient part of the total cost being considered and the decision between alternatives must be close enough so that a revision in the estimate of the life distribution, and consequently R_{cr}, could conceivably reverse the results of the analysis.

In any case, if there is uncertainty concerning the distribution of life but the coefficient of variation is known to be low, e.g., below 10 per cent, then there would be little effect caused by this lack of certainty, because R_{cr} is very close to 1.00 for all coefficients of variation between 0 per cent and 10 per cent. On the other hand, if there is uncertainty concerning the life distribution such that there could be appreciable changes in the probabilities that the life will turn out to be short, then cause for concern as to the correctness of the estimated life distribution may exist, for R_{cr} can be affected considerably by changes of this nature. The next chapter contains a method to facilitate the consideration of lack of certainty when project life is the only element for which variation is taken into account.

Lack of Certainty of Multiple Elements--Overall Effect on Analysis Outcome

In this section consideration will be made of the effects of lack of certainty when dispersion of multiple elements is being considered. Chapter IV presented the results of a sensitivity study which showed the relative change in PV($) caused by changes in each of four major elements. Chapter IV also contained computational means for approximating the expected value and variance of the distribution of PV($) through estimates of the expected value and variance of each of these major elements.

Just how much difference in the analysis outcome can be caused by lack of certainty concerning the distribution of an individual element in that analysis depends upon the circumstances inherent in the analysis. The major circumstances affecting the importance of that lack of certainty are: the relative importance of each element to the overall analysis outcome due to the magnitude of the element and sensitivity to the element, the closeness of the decision based on initial estimates of the respective element distributions, the degree of lack of certainty concerning those distributions, and the effects of variation of combinations of elements.

Figures 52-57 can be used for visual comparison of the relative effects of various element magnitudes and amounts of variation on overall analysis outcomes, thus aiding in the exploration of uncertainty concerning those quantities. They apply only to mutually exclusive projects for which net disbursements, rather than net receipts, are known. In these figures, the relative effects of different proportions of element magnitudes are taken into account by considering ratios of disbursements to capital recovery cost (denoted C.CRC($)) of 0, 0.5, 1.0, and 2.0; salvage values of 0 per cent and 40 per cent; and expected lives of 4, 8, and 16 years. All combinations of these element magnitudes are considered for interest rates of 5 per cent, 10 per cent, and 20 per cent.

The effects shown in Figures 52-57 are for both the expected value and the standard deviation of the measure of merit expressed in AW($). To attain generality, these effects are shown as ratios--both are divided by the AW($) computed for assumed certainty (denoted as C.AW($)). The relative effect of independent simultaneous variation of project investment, life, salvage value, and periodic disbursements, each having the same coefficient of variation, is shown across the top row of each figure. For comparison purposes, the relative effect of variation of life only is shown across the bottom row of each figure. All results are shown for coefficients of variation between 0 and 30 per cent for each of the element(s) varying. Since the effect on E(AW($))/C.AW($) for a given set of conditions is the same regardless of whether all elements are varying or only life is varying, this ratio is shown only in the top row of each figure.

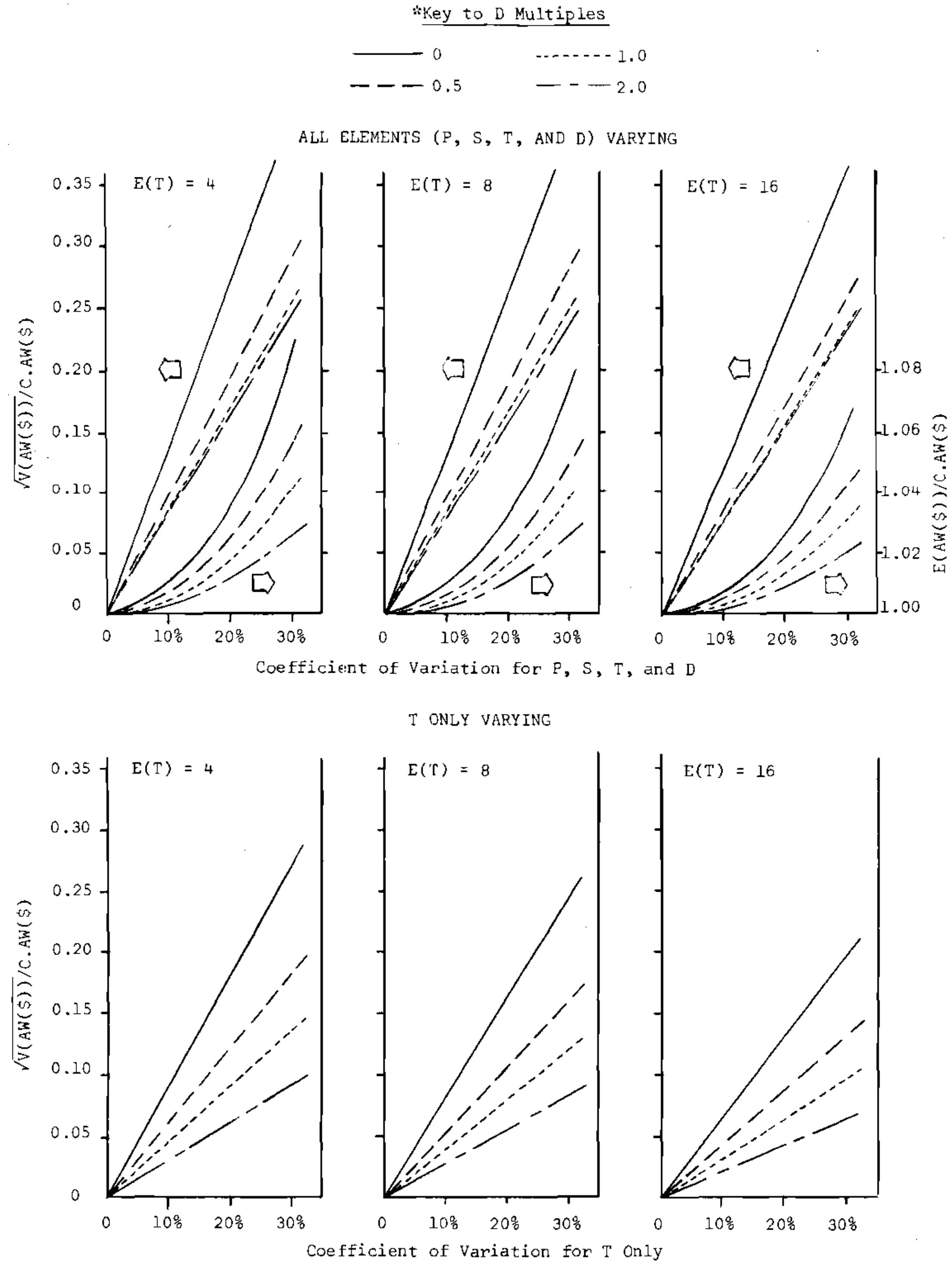


Figure 52. Relative Effect of Variation on E(AW($)) and $\sqrt{V(AW(\$))}$
[For D as Various Multiples of C.CRC($)]*
5% Interest, 0 Salvage

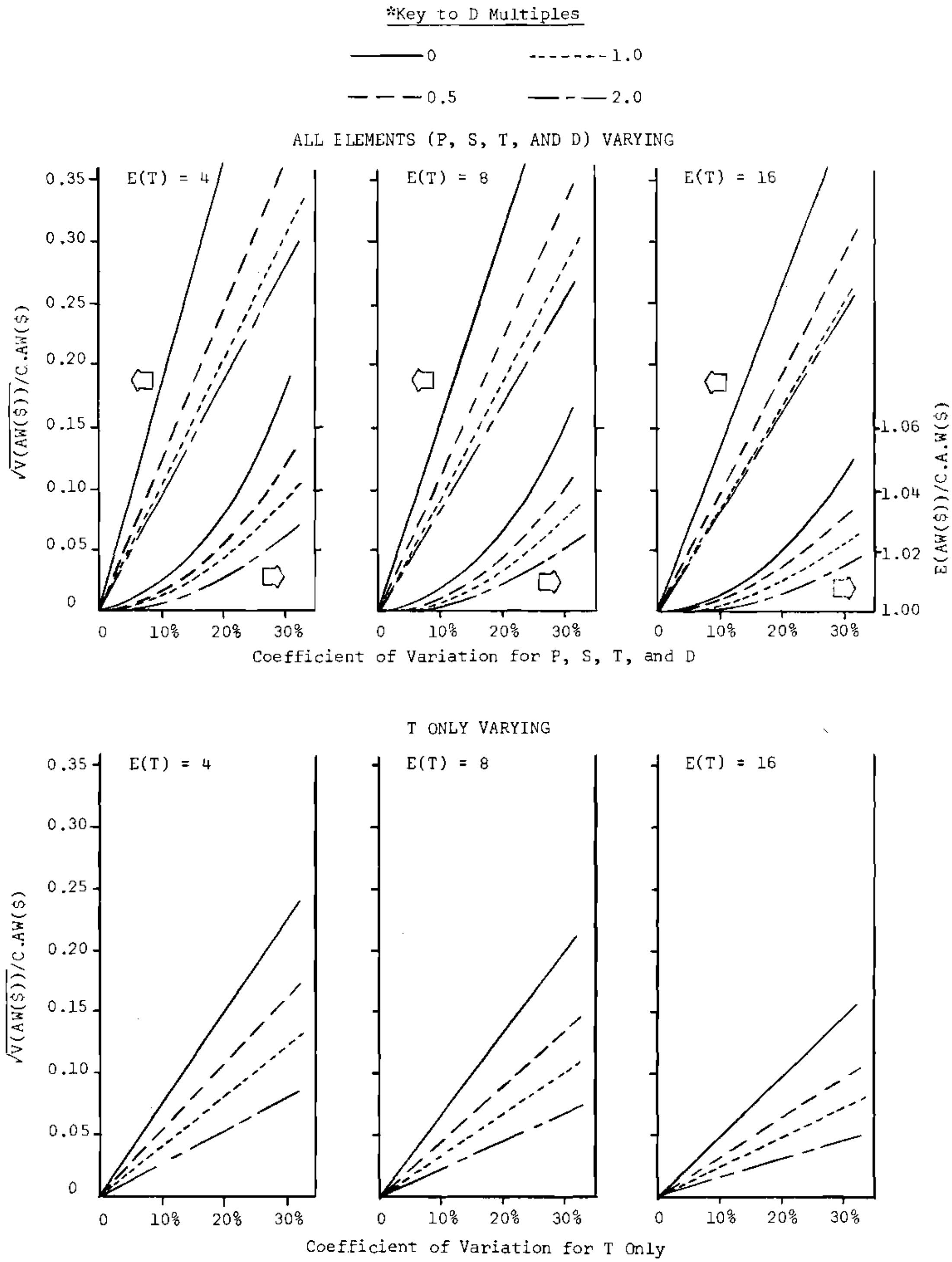


Figure 53. Relative Effect of Variation on E(AW($)) and $\sqrt{V(AW(\$))}$ (For D as Various Multiples of C.CRC($))*
5% Interest, 40% Salvage Value

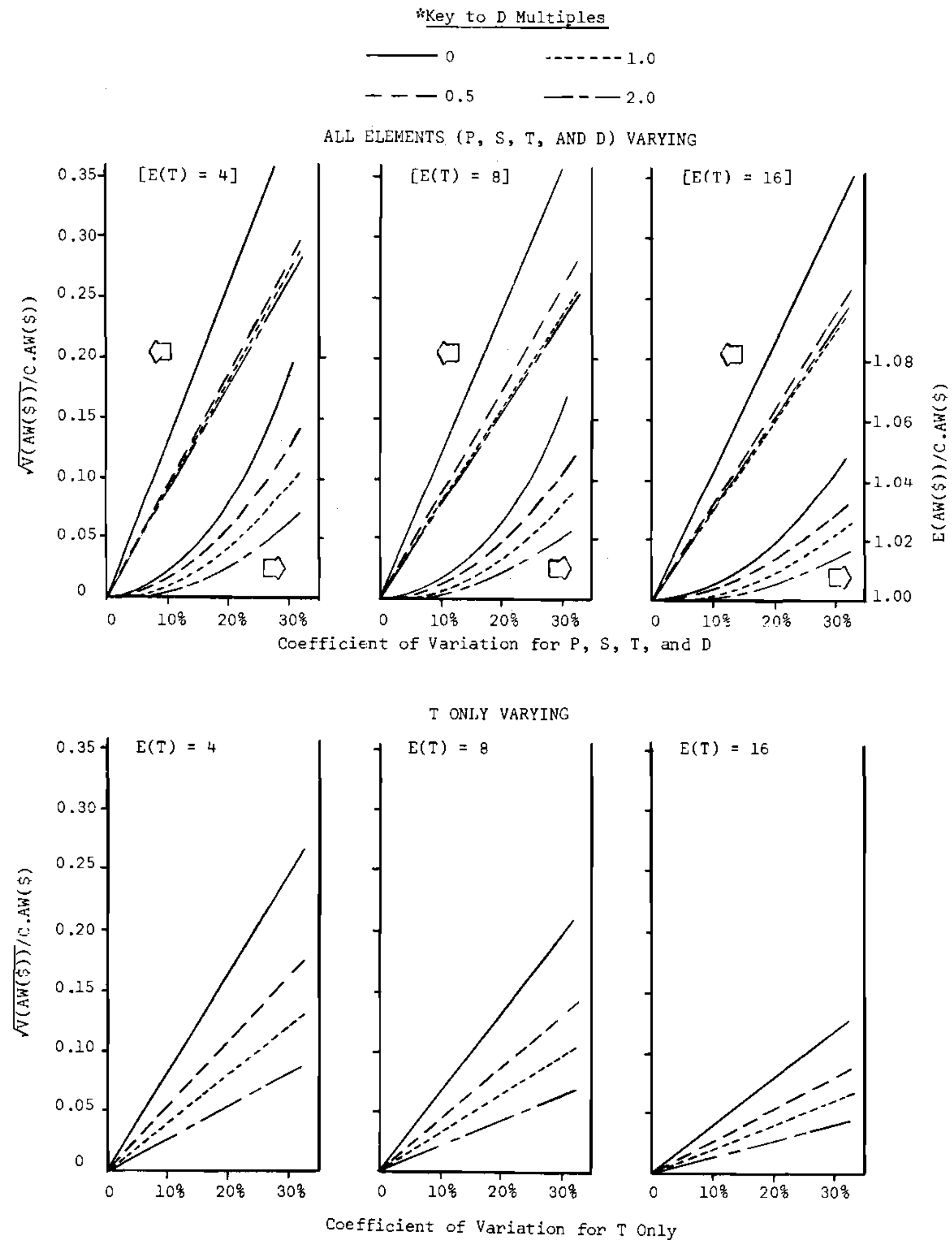


Figure 54. Relative Effect of Variation on E(AW($)) and √V(AW($))
[For D as Various Multiples of C.CRC($)]*
10% Interest, 0 Salvage Value

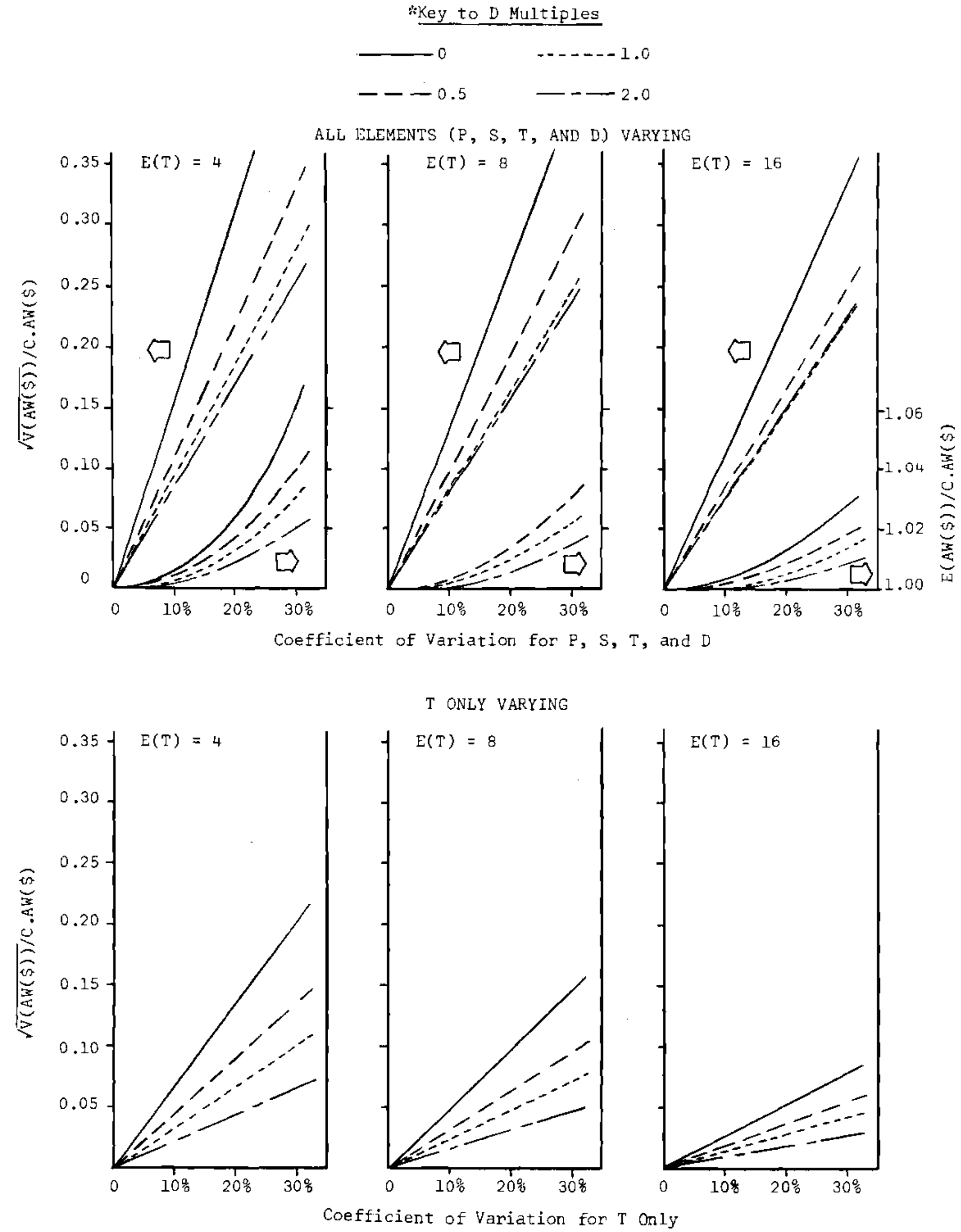


Figure 55. Relative Effect of Variation on E(AW($)) and $\sqrt{V(AW(\$))}$
[For D as Various Multiples of C.CRC($)]*
10% Interest, 40% Salvage

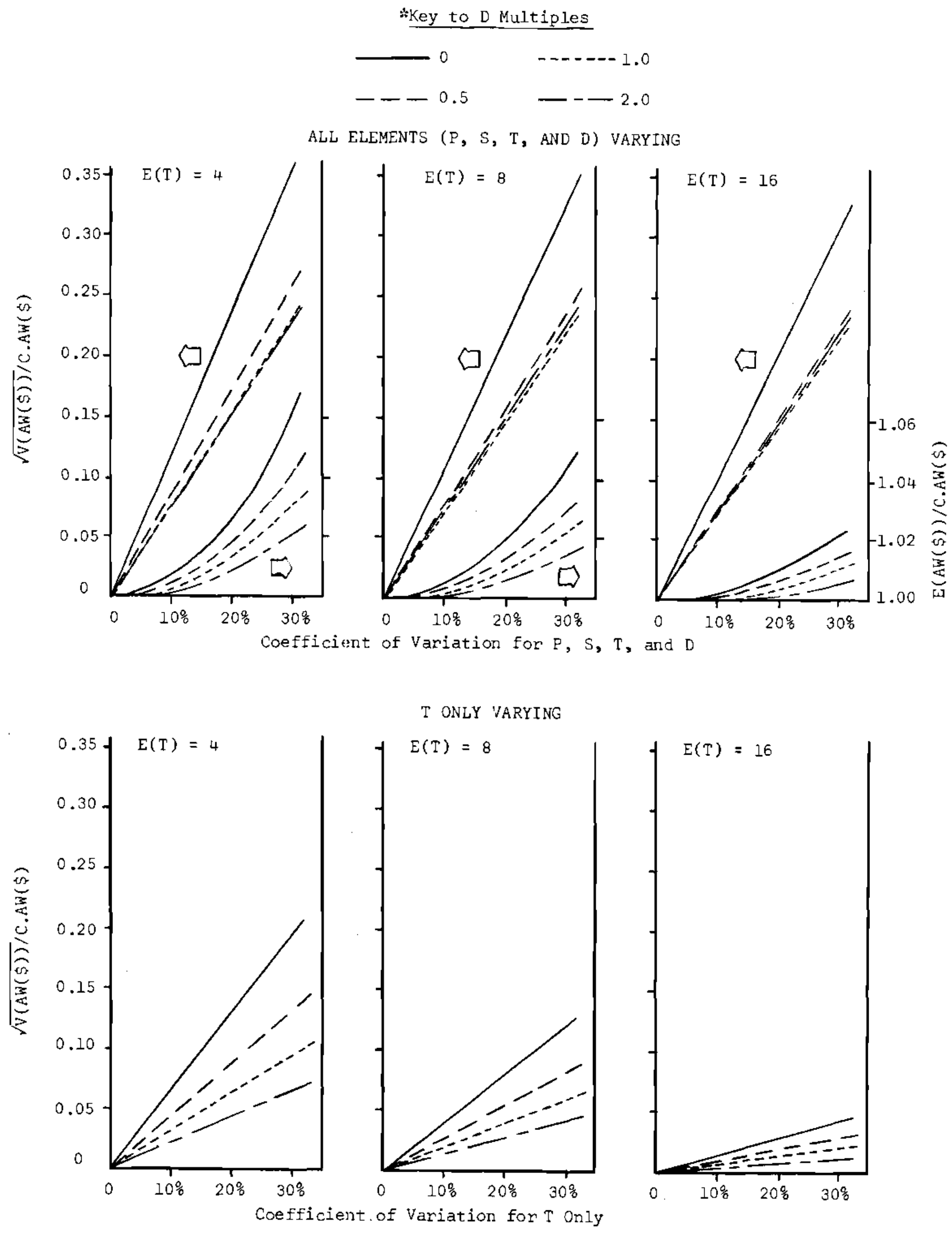


Figure 56. Relative Effect of Variation on E(AW($)) and $\sqrt{V(AW(\$))}$
[For D as Various Multiples of C.CRC($)]*
20% Interest, 0 Salvage

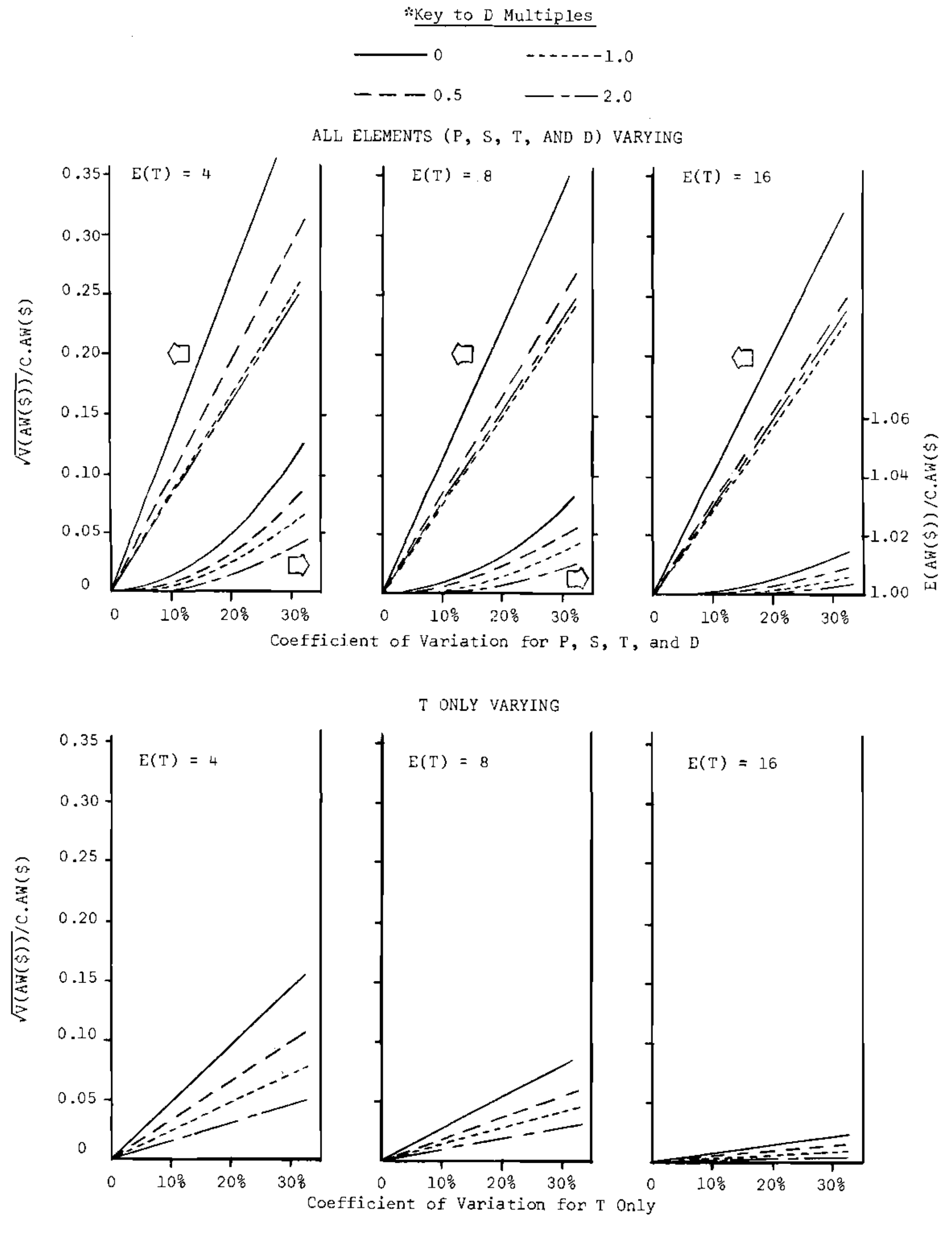


Figure 57. Relative Effect of Variation on E(AW($)) and $\sqrt{V(AW(\$))}$
[For D as Various Multiples of C.CRC($)]*
20% Interest, 40% Salvage

Table 19 in Appendix G shows the computer program for calculations used to obtain Figures 52-57. E(AW($)) and V(AW($)) are approximated through use of the Taylor series expansion.

Figures 52-57 include only a limited number of conditions of variation of individual elements, but the range of element values considered is fairly broad. These figures, together with parallel figures if needed for other conditions, can be used as a quick aid in judging effects of lack of certainty on analysis outcomes. The next section discusses decision guides concerning the extent of economic analyses.

Decision Guides on When and How Much to Consider Random Variation of Multiple Elements

A key question in making economic analyses is: Under what circumstances is it worthwhile to quantitatively consider random variation of elements in an economic analysis; and, when considered, to what extent should that consideration be pursued? A conceptual answer to this question boils down to simple economics: Put more study effort into the analysis as long as the savings from further study is greater than the cost of further study (i.e., as long as marginal savings is greater than marginal cost). Since the marginal savings (and possibly marginal cost also) for a given amount of added study is a variable, it is necessary to modify the rationale. A reasonable modification seems to be the consideration of expected values. Thus, the rationale can be restated as: Put more study effort into the analysis as long as the expected savings from further study is greater than the expected cost of that further study.

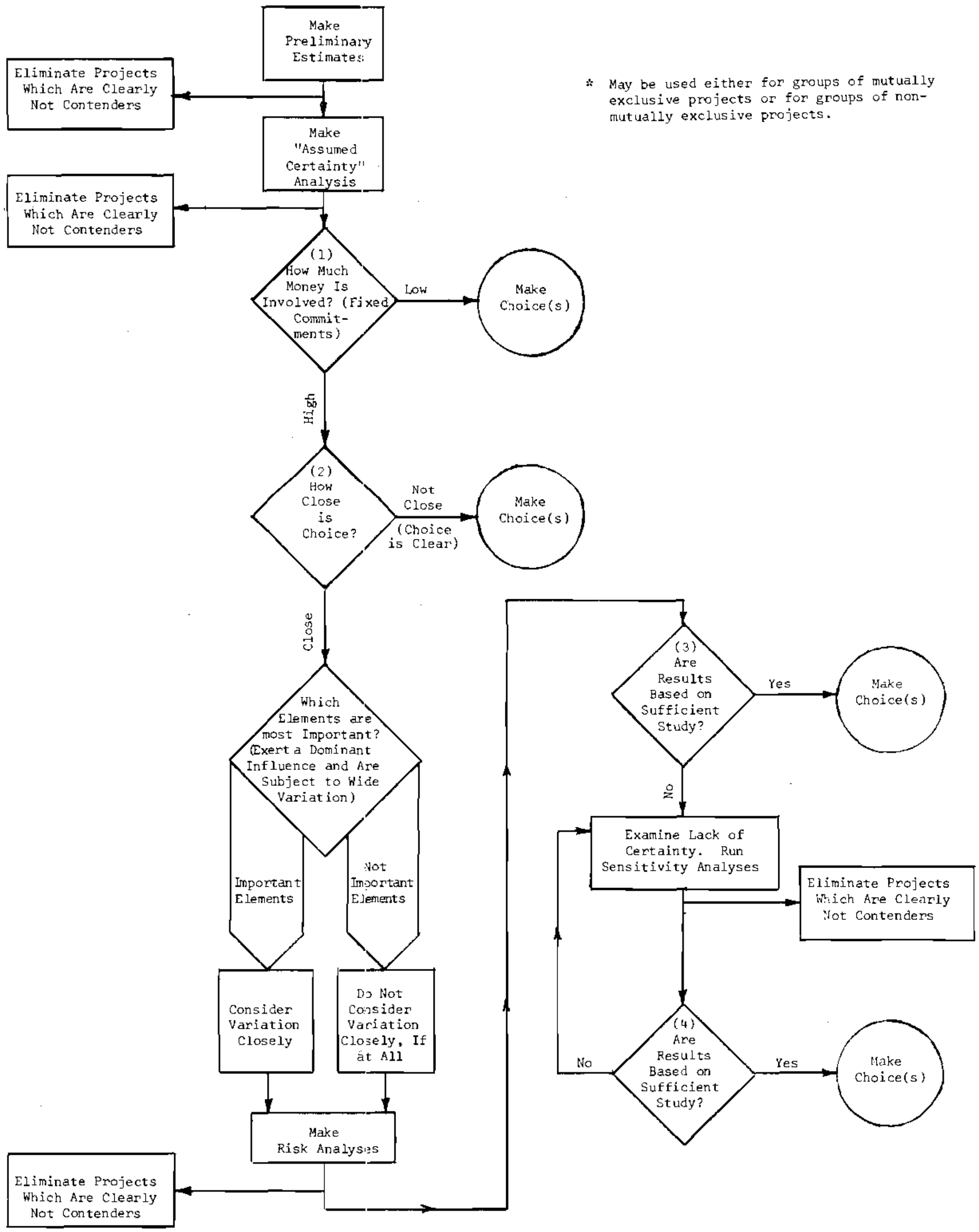
The great problem in applying this rationale in practice is that it is quite difficult to estimate the expected savings from further study. In economic analyses of mutually exclusive projects, savings from further study occur if the further study correctly causes a reversal in the project accepted. In economic analyses of non-mutually exclusive projects, savings from further study occur if the further study correctly causes the decision maker to drop one or more projects previously accepted. Other savings can be created by the added study. For example, the added study may provide information which will prove useful in future operating decisions and/or investment analyses.

The savings from further study can be conceptually determined as the discounted present value of the new project(s) accepted after the further study minus the present value of the project(s) accepted before the further study and dropped as a result of the study. However, the practical problem of determination of the expected savings from further study, as based on the amount of savings and the likelihood or probability of those savings, would generally be quite difficult. It should be noted that the expected savings from added study may well not be a continuous function of the amount of the added study, but rather it is likely to change in discrete steps.

The expected cost of added study is more readily determinable than the expected savings from that study, but nevertheless it is not always apparent. Two common viewpoints on this cost are that it is equal to the direct cost of the resources devoted to the added study or that it is essentially zero on the grounds that the resources are available and paid for regardless of whether they are used on that added study. The most defensible cost of added study is based on the opportunity cost principle. That is, the cost of the added study should be determined by the value to the company of those study resources if put to best productive use on work other than that added study. While this opportunity cost is often hard to evaluate, it seems reasonable that in a well-managed company the cost will be at least as great as the direct cost of those resources.

Figure 58 shows a flow diagram which depicts a recommended sequence of steps in making economic analyses and shows qualitative test points regarding the extent of the analysis. This sequence would be applicable to analyses of either groups of mutually exclusive or non-mutually exclusive projects. Note that the recommended sequence shown in the figure shows four different points at which the decision could be made concerning which project(s) to accept. Also, there are four stages at which provision is made for dropping from further consideration projects which analysis indicates are clearly not contenders worthy of further study.

The meaning of the test points included is worthy of discussion. The test points are depicted as diamond shapes and are numbered in parentheses. Test point (1) considers the magnitude of the money involved in the decision for purposes of deciding whether further study is justified. The relevant amount of money to consider is the total present value of the fixed commitments associated with each project. These fixed commitments would include the non-recoverable investment costs as well as other fixed costs which the company would incur if it should accept that project. If the magnitude of the fixed commitments for each of the projects being considered is low compared to the cost of further study, then



* May be used either for groups of mutually exclusive projects or for groups of non-mutually exclusive projects.

Figure 58. Recommended Sequence of Steps for Economic Analyses Showing Decision Points on Extent of Analysis*

it may be decided that further study is not justified and that the choice(s) should be made. The breakeven point concerning the size of fixed commitments to use as this criterion is rather subjectively determined. Intuitively, it appears that this point would be related to the company's financial health, the size of the projects usually considered, and the availability of resources for further analysis.

Test point (2) in Figure 58 considers how close is the choice between projects. If the assumed certainty analysis results up to that point show that the decision is not at all close (i.e., the choice(s) are apparent), then further study is hardly justified and the choice(s) should be made.

Test point (3) is concerned with the decision of whether the results of an initial analysis considering variation of elements (which would be essentially a risk analysis) is based on sufficient study considering the economic importance of the decision and the closeness of the analysis results for the projects considered. If the decision is important enough in terms of worth of the fixed commitments for the projects considered and the analysis results are somewhat close, then further study should be performed before the choice(s) are made. The further study would take the form of closer estimations of elements and sensitivity analyses.

Test point (4) is repetitious of test point (3). It shows that closer estimations and sensitivity analyses would be continued until it is decided that the results of the analyses are based on sufficient study and the choice(s) can be made.

The sequence suggested in Figure 58 is subject to short cuts in cases where warranted. For example, if a given analysis involves projects that are of extreme importance to the future of the company, it may be decided to directly perform a risk analysis which considers variation of multiple elements without bothering to perform an initial "assumed certainty" study.

The value of the sequence of steps shown in Figure 58 is that it provides a conceptual basis for determining the extent to which economic analyses should be performed. The decisions in the sequence are rather intangible, but nevertheless the sequence of steps represents a formalization of the thinking which the analyst should perform in determining the extent to which analyses should be carried out in particular situations.

CHAPTER VII

EXTENSIONS OF METHODS FOR CONSIDERING RANDOM VARIATION OF ELEMENTS IN CAPITAL INVESTMENT ANALYSES

Introduction

This chapter contains miscellaneous methods and techniques proposed as potentially useful for considering random variation of important elements in capital investment analyses. These methods are disjoint and of limited applicability. However, they are extensions to methods and approaches which have been presented in the literature and may, upon occasion, aid the analyst in formally taking variation of elements into account.

Computer Program for Sensitivity Studies

Perhaps the most useful of the extensions of methods shown herein is a special computer program developed for ease in performing sensitivity studies for individual projects. The program shown herein is unique because it provides measures of analysis outcomes when one or more elements vary from their respective expected values by the same relative amounts. The elements for which variation can be considered are P, S, T, and D.

The program first provides for calculation of approximations of $E(AW(\$))$ and $\sqrt{V(AW(\$))}$ by use of the Taylor Series expansion as described in Chapter IV. The calculation of $E(AW(\$))$ is quite accurate for most economic analysis purposes because of the fact that it includes the first three terms of the expansion. $\sqrt{V(AW(\$))}$ as calculated in the program is based only on two terms of the Taylor series expansion so as to include the consideration of correlation only between project life and periodic operating results, and between project life and salvage value.

The sensitivity study part of the program consists of provision for determining the effect on AW($) when each element in any combination of one or more elements varies a certain number of standard deviations in a favorable or unfavorable direction from the expected values of those elements. The resulting AW($) values reflect rather extreme optimistic or pessimistic results. The analysis results reflecting the effects of elements varying in an unfavorable direction can be quite useful to the decision-maker for judging the severity of the effect on AW($) when adverse element outcomes occur in combination.

The program is designed so that the number of standard deviations, σ, of variation for each element which is subject to variation can be indexed between any two extreme amounts in any desired discrete increments. Table 16 in Appendix G shows the program written in ALGOL so that the number of σ's of variation considered is between -2.0 and +2.0 in increments of 0.5. The program as shown in Table 16 includes provision for the elements P, S, D, and T to all vary simultaneously in the sensitivity study. If it is desired to make the study without considering variation of one or more of these elements, merely remove from the program input information corresponding to each element for which variation is not to be considered. Table 9 shows the input information corresponding to each element for which variation can either be considered or not be considered.

Table 9. Input Information Corresponding to Optional Elements--Sensitivity Program

Element	Input Information
P	KP = P + K•SP
S	KS = S + K•SS
D	KD = D + K•SD
T	KT = T + K•ST

<u>Example Problem Using Program</u>

As an example of the type of information which can be obtained from this special program for performing sensitivity studies, suppose two projects, A and B, are being compared. For project A, the expected values for each of the elements considered are: P = -$100,000, S = $10,000, D = $15,700, and T = 10 years. For project B, the expected values for each of the elements considered are the same as for project A, except that D = $17,000. Interest on invested capital is 10 per cent. For purposes of calculating V(AW($)) for each project, all elements are assumed to vary independently. Normally, there would need to be individual estimates of the magnitude of σ for each element for which variation is considered. For purposes of this example, the coefficient of variation for each element will be assumed to be 20 per cent for project A and 30 per cent for project B. It is desired to explore the sensitivity of AW($) to the elements P, S, T, and D varying from their respective expected values according to the program in Table 16. Table 10 shows the results from use of this computer program for the above conditions.

Table 10. Results for Example Problem--
Computer Sensitivity Program

OVERALL RESULTS

	Project A	Project B
E(AW($))	$ 104	$ 955
$\sqrt{V(AW(\$))}$	4,750	7,400

SENSITIVITY RESULTS (IN AW($))

Variation of Each Element (No. of σ's from Respective Expected Values)	Project A	Project B
-2.0	$-20,879	$-40,919
-1.5	-14,143	-24,175
-1.0	- 8,579	-13,233
-0.5	- 3,790	- 5,000
0.0	462	1,762
0.5	4,328	7,644
1.0	7,909	12,935
1.5	11,245	17,909
2.0	14,474	22,593

Note from the above results how AW($) varies for the different conditions considered. Project B has a higher E(AW($)) than project A. However, for the condition of extreme variation of all elements in an unfavorable direction, project B shows about twice as great an annual loss as project A. On the other hand, for the condition of extreme variation of all elements in a favorable direction, project B shows about 50 per cent greater positive AW($) than does A.

Use of Expected Opportunity Loss Information to Value Decreased Lack of Certainty

A technique which can be useful in judging the extent to which consideration of random variation should be made is a particular use of "expected opportunity loss" or "expected cost of uncertainty" information. The concept and computation of the expected opportunity loss, denoted EOL, is credited to Schlaifer (29) and is discussed in the section on "Supplementary Criteria" in Chapter V. The technique proposed here involves using EOL information to aid in placing a value on decreased lack of certainty when that decrease can be obtained through added expenditure of resources or effort in the analysis.

Conceptually, when the analyst is relatively unsure of the outcome for a project, the estimated distribution of outcomes for that project would have a relatively large variance. If, through added effort in the analysis, the analyst is more certain of the outcome, then the estimated distribution of outcomes for that project would be characterized by a decreased variance. If, for a given expected outcome, the variance is decreased, then the expected opportunity loss is decreased. This decrease in EOL can be used as a guide to measuring the value of that added analysis effort.

As an example of the use of this technique, consider the illustration near the end of Chapter V in which a certain amount of analysis effort results in $\hat{E}(PV(\$)) = \$4{,}000$ and $\sqrt{\hat{V}(PV(\$))} = \$8{,}000$, where PV($) is normally distributed. For those conditions, the EOL = $1,582. Suppose that the analyst makes an "a priori" estimate that a given amount of added analysis effort would reduce $\sqrt{V(PV(\$))}$ to $6,000, and that the distribution of PV($) would still be normal with $\hat{E}(PV(\$)) = \$4{,}000$. With the new lowered $\sqrt{\hat{V}(PV(\$))}$ the EOL = $6,000 x (unit normal loss integral at $\frac{\$4{,}000}{\$6{,}000}$)[8] = $6,000 x 0.151 = $906. While the decrease in EOL = $1,582 - 906 = $676 is not an actual savings, it is a measure of the decrease in the expected opportunity loss due to uncertainty which can be attributed to the added analysis effort and can thus be useful in placing a value on that added analysis effort. The value of that added effort can then be compared with the cost of that added effort so that a decision can be made on the advisability of the added analysis effort.

If the analyst is judging the value of more than one level of added analysis effort, he might use the technique shown above to estimate the decrease in the expected opportunity loss for each higher level of effort. These results can be compared with the cost of each higher level of effort in order to determine which level of added effort seems to be optimal. Conceptually, one would expend added analysis effort as long as the marginal decrease in the EOL has greater value than the marginal cost of that added effort. To aid the analyst in easily determining the effect of changes in parameters of a normal distribution on the expected opportunity loss, Figure 59 is provided.

Increasing Variance Functions to Reflect Increasing Future Lack of Certainty

In general, the more distant the future time for which an estimate is made, the greater the lack of certainty in that estimate. There

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8. See Appendix F for further explanation of formula.

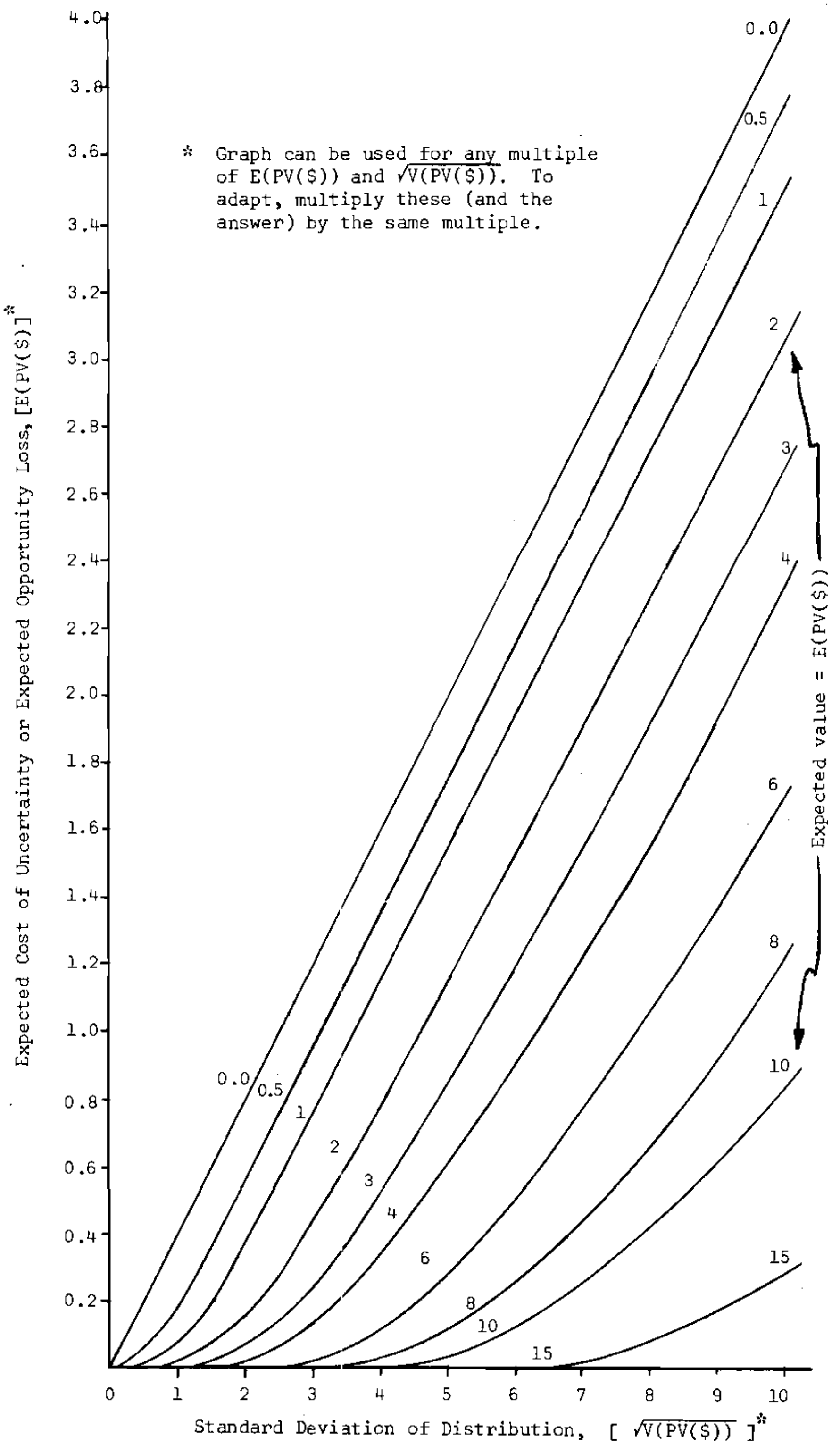

Figure 59. Effect of Various Expected Values and Standard Deviations on Expected Opportunity Loss for Normal Distribution

are numerous suggestions in the literature pertaining to reflecting the greater uncertainties coincident with increasing future time by the use of higher discount rates. This is applicable to assumed certainty studies. In cases where the distribution of outcomes for an element or project is considered, one can generally reflect an increasing lack of certainty coincident with more distant future time by estimating increasing variances for that distribution. In the case of estimates of a continuing series of cash flows, it might be convenient and appropriate to estimate a mathematical function according to which that variance is expected to increase with time. Some typical mathematical functions are given below:

If the variance increases linearly from σ_o^2 at time 0 to σ_T^2 at time T, then the variance, σ_t^2, at any time, t, between 0 and T can be written as:

$$\sigma_t^2 = \sigma_o^2 + \frac{t}{T}(\sigma_T^2 - \sigma_o^2) \ . \qquad (52)$$

As a simplification, let

$$\left[\frac{(\sigma_T^2 - \sigma_o^2)}{T}\right] \div \sigma_o^2 = \text{rate of linear increase} = \Delta \ .$$

Then,

$$\sigma_t^2 = \sigma_o^2 + (t)(\Delta)(\sigma_o^2) \ . \qquad (53)$$

If the variance is assumed to increase at a constant proportion, δ, of the variance in the previous period, this can be shown as:

$$\sigma_t^2 = (\delta)(\sigma_{t-1}^2) \ . \tag{54}$$

If the variance is assumed to increase exponentially from σ_o^2, this can be written as:

$$\sigma_t^2 = (\sigma_o^2)(e^{kt}) \tag{55}$$

where k is the coefficient (positive) corresponding to the applicable rate of exponential increase.

The various functions for reflecting increasing future lack of certainty as shown in Equations (52) through (55) were all based on the variance. They could just as well have been based on the standard deviation or some other measure of dispersion. This method could be useful in obtaining increased accuracy in the distributions of outcomes developed in analyses, because it facilitates explicit consideration of the effect of increasing future time on those distributions.

A Method for Considering Lack of Certainty Concerning the Distribution of Project Life

In this section, a method will be shown for aiding the analyst in considering the distribution of project life when there is lack of certainty concerning that distribution and all other elements are assumed to be known. The method is essentially a breakeven analysis where the breakeven point is a value of R_{cr} at which a project is marginally acceptable. This method will be developed for both single project analyses and for analyses involving the comparison of projects.

Single Projects

When project life is the only element for which dispersion is considered, the expected annual worth for that project may be expressed as:

$$E(AW(\$)) = (P+S)(CRF)(R_{cr}) - S(i) + D \qquad (56)$$

A single project, by itself, can be justified only when receipts, D, are known and the net AW($) is positive or zero. If the project is just barely acceptable, E(AW($)) is zero. Thus, AW($) is just barely acceptable if R_{cr} is equal to a particular value, R_{cr}^*, where

$$R_{cr}^* = \frac{S(i) - D}{(P+S)(CRF)} \; . \qquad (57)$$

If the R_{cr} which the analyst judges to be applicable for the project under consideration is greater than R_{cr}^* as calculated above, then the project is not acceptable. Conversely, if the R_{cr} which the analyst judges to be applicable is less than R_{cr}^*, then the project is acceptable.

As an example, suppose a project is being evaluated for which the following estimates apply: P = -$100,000, S = $10,000, D = $15,700, i = 10 per cent, and T = 10 years. Calculation of the value of R_{cr}^* shows:

$$R_{cr}^{*} = \frac{\$10{,}000(0.1) - \$15{,}700}{(-\$100{,}000+\$10{,}000)\cdot((0.1)/1-e^{-(0.1)(10)})} = 1.03$$

Thus, if the R_{cr} which is expected to be applicable is equal to or less than 1.03, then the project is acceptable. Otherwise, it is unacceptable.

Note that the decision on the acceptability of the project can be made by calculating R_{cr}^{*} without bothering to calculate AW($) under assumed certainty. (As a side note, the assumed certainty AW($) for the example project is $462.) If the calculated value of R_{cr}^{*} is obviously higher than the R_{cr} which is estimated to be applicable, then the project would be acceptable without further question. If the value of R_{cr}^{*} is obviously lower than the R_{cr} which is estimated to be applicable, e.g., below 1.00, then the project would be unacceptable without further question. If the value of R_{cr} is in some intermediate range, then the analyst is forced to make a close estimate of the dispersion of life expected so that he can determine a value of the applicable R_{cr} to compare against R_{cr}^{*} and thus decide if the project is acceptable. The determination of the R_{cr} which is applicable for given conditions of life dispersion and interest rate can be facilitated by the use of graphs such as in Figures 6-17.

Comparison of Two Projects

Paralleling the development of the last section, consider the case in which choice is to be made between two mutually exclusive projects. The annual worths for two projects, A and B, are equal when:

$$(P_A+S_A)(CRF_A)(R_{crA})-S_A(i)+D_A = (P_B+S_B)(CRF_B)(R_{crB})-S_B(i)+D_B. \qquad (58)$$

The subscripts A and B refer to the project designations. By manipulation of Equation (58), it can be found that the value of R_{crA}, R_{crA}^*, at which the projects are equally economical is:

$$R_{crA}^* = \frac{-S_B(i) + S_A(i) + D_B - D_A + (P_B + S_B)(CRF_B)(R_{crB})}{(P_A + S_A)(CRF_A)}. \qquad (59)$$

It can be observed that this relation requires all the elements of an assumed certainty study and, in addition, an estimated or assumed value of R_{crB}.

If the R_{crA} which is judged to be applicable is greater than R_{crA}^*, then A would not be the best project, and B would be the choice. Conversely, if the R_{crA} which is judged to be applicable is less than R_{crA}^*, then A rather than B would be the project which should be selected.

The same method as above can be used to find the value of R_{crB}, R_{crB}^*, at which the two projects are equally economical. Thus:

$$R_{crB}^* = \frac{-S_A(i) + S_B(i) + D_A - D_B + (P_A+S_A)(CRF_A)(R_{crA})}{(P_B+S_B)(CRF_B)}. \qquad (60)$$

Note that this method for finding R_{crB}^* requires an estimate of R_{crA}. The criterion for project selection using R_{crB}^* parallels the criterion when using R_{crA}^* as explained above.

This breakeven type of analysis for comparing two projects has the advantage that it allows one to take into account the effect on expectations caused by consideration of life dispersion without necessarily having to make a close estimate of R_{cr} for one of the projects.

Consideration of Different External Conditions in Estimating the Distribution of Result Value[9] for an Element or a Project

This section contains a suggested system which can facilitate increased accuracy of estimates for economic analyses. This system is appropriate for use when the result value of an element or a project depends distinctly on the outcome of some external or "state of nature" condition. Examples of external conditions which might have direct influence on the result value of a project are general business conditions, the amount of new competition, the extent of a potential steel strike, etc. The system involves breaking the estimation process down into separate estimates of the conditional mean and variance of the result value for each outcome and then combining these estimates with probabilities of each outcome to calculate the estimated mean and variance of the overall result value. General mathematical formulation is shown in Appendix A. The main results are:

$$E(x_i) = \sum_j f_j(0_j) \cdot E(x_i | 0_j), \tag{61}$$

9. In this section and in the last section of this chapter, "result value" will be used to mean the same as "outcome" or "analysis outcome" as used in the remainder of this work. In these last two sections of this chapter, "outcome" has particular meanings.

and

$$V(x_i) = \sum_j f_j(0_j)\cdot V(x_i|0_j) + \sum_j f_j(0_j)\cdot\{E(x_i|0_j)\}^2 - [E(x_i)]^2 , \qquad (62)$$

where 0_j ~ j^{th} outcome or state of nature,

x_i ~ i^{th} result value,

$f_j(0_j)$ ~ density function of outcomes,

$E(x_i|0_j)$ ~ expected result value if j^{th} outcome occurs,

and

$V(x_i|0_j)$ ~ variance of result value if j^{th} outcome occurs.

As an example of the use of this system, suppose that it is desired to determine the expected value and variance of the result value, say PV($), for a project where the outcomes or states of nature to be considered are business conditions. There are three different business conditions considered, and the relevant probabilities and means and variances of the result values for each outcome are shown in Table 11. Solutions to obtain estimated $E(x_i)$ and $V(x_i)$ for the example are:

$$E(x_i) = 0.3(100) + 0.5(50) + 0.2(-20) = \underline{\underline{51}}$$

$$V(x_i) = 0.3(6400) + 0.5(3600) + 0.2(4900) + 0.3(100)^2 + 0.5(50)^2 + 0.2(-20)^2 - (51)^2 = \underline{\underline{6,390}}$$

A breakdown of the estimation process such as illustrated above intuitively should result in estimates of $E(x_i)$ and $V(x_i)$ which are more accurate than if the estimates are made directly without that breakdown.

Table 11. Data for Example--Consideration of External Conditions in Estimation

Parameters of Conditional Result Values	OUTCOME, BUSINESS CONDITIONS (O_j) Good(O_1) $f_1(O_1) = 0.3$	Average (O_2) $f_2(O_2) = 0.5$	Poor (O_3) $f_3(O_3) = 0.2$
$\hat{E}(x_i \mid O_j)$	100	50	-20
$\hat{V}(x_i \mid O_j)$	6,400	3,600	4,900

Adaptations of Decision Rules for Complete Uncertainty to Continuous Outcome Problems

This section describes adaptations of existing arbitrary decision rules for complete uncertainty. These existing rules[10] apply to situations where the outcomes are discrete and there is complete uncertainty concerning the probabilities of occurrence for each possible state of nature or outcome. However, the existing rules, as developed, do assume certainty concerning the result values for all alternatives and outcomes.

In these adaptations to continuous outcome problems, complete uncertainty is assumed as to the outcome (in terms of number of standard deviations of fluctuation from the expected result value) which will occur. However, certainty is assumed concerning the mean and variance of the distribution of the result values for each project under considera-

10. A good reference on the existing rules is Morris (24-Chapter 17). They were described in the last part of Chapter II.

tion, but no assumption is made concerning the form of the distribution.

In order to use these adaptations, the analyst must arbitrarily decide on the amount of fluctuation of the outcomes to be considered. The magnitude of fluctuation chosen (expressed in standard deviations) should represent the extreme outcomes of concern. Since $\pm 2\sigma$ would encompass a very large proportion of the density of most distributions of interest,[11] it will be used to demonstrate the use of these adapted decision rules under conditions of rather extreme fluctuations (outcomes). To demonstrate the effect of somewhat intermediate fluctuations (outcomes), parallel results using $\pm 1\sigma$ will also be shown.

As an example, consider in Table 12 three mutually exclusive projects together with information on the means and standard deviations of the result values, expressed in PV($), for each project.

Table 12. Data for Example--Adapted Decision Rules for Complete Uncertainty

Project	$\hat{E}(PV(\$))$	$\sqrt{\hat{V}(PV(\$))}$
A	20	10
B	11	2
C	18	20

11. According to Tchebyscheff's inequality, if the distribution has a finite variance, the least possible proportion of the density that could be within the $\pm 2\sigma$ limits is 0.75. By the Camp-Meidell inequality, if the distribution is unimodal, that minimum possible proportion is increased to 0.889. If the distribution is normal, the proportion is increased to ± 0.9545.

Table 13 shows the result values for various outcomes (amounts of fluctuation) to be considered in this example. The information shown in this table is analogous to the information used for the existing arbitrary decision rules.

Table 13. Result Values for Various Outcomes in Example--Adapted Decision Rules (Result Values in PV($))

Project	Outcome (Number of Standard Deviations from Mean) -2σ	-1σ	1σ	2σ
A	0	10	30	40
B	7	9	13	15
C	-22	-2	38	58

Consider first the outcome fluctuations of $\pm 2\sigma$. If the adaptation of the maximin (conservative) rule were applied, one would observe that for the outcome resulting in minimum PV($) for all projects, -2σ, project B has the maximum PV($), 7. Hence, B would be the project selected. If the adaptation of the maximax (optimistic) rule were applied, the outcome which results in maximum PV($) for all projects is $\pm 2\sigma$. For this outcome, project C has the maximum PV($), 58, and would be the project selected. If the adaptation of the Hurwicz principle were applied, the weighted result value depends upon the index of optimism. Table 14 below shows sample computations for an index of optimism of 0.2 only. By examination of Table 14, it can be seen that projects B, A and C would be the choices for indices of optimism of 0.2, 0.5, and 0.8, respectively.

Table 14. Sample Computations and Weighted Result Values--Adapted Decision Rule Using Hurwicz Principle (Result Values in PV($))

Project	Index of Optimism 0.2	0.5	0.8
A	0.2(40) + 0.8(0) = 8.0	20.0	32.0
B	0.2(15) + 0.8(7) = 8.6	11.0	13.4
C	0.2(58) + 0.8(-22) = -4.8	18.0	42.0

If these above adaptations of the arbitrary decision rules were used considering the more moderate fluctuations (outcomes) of $\pm 1\sigma$, many of the answers would be changed. For example, using the adapted maximin rule, project A would be the choice. Using the adapted maximax rule, project C would be the choice. Using Hurwicz principle, projects C, A, and B would be the choices for indices of optimism of 0.2, 0.5, and 0.8, respectively.

These adapted decision rules are admittedly arbitrary as are the uncertainty decision rules after which they are patterned. However, they do have the advantage of promoting consistency of choice.

It is not claimed that any of the methods and techniques presented in this chapter are profound breakthroughs in the technology of considering risk and uncertainty in capital economic analyses. However, they do represent extensions to the present literature, and seem to be worthy of consideration for application in practice. The next chapter is concerned with the use of statistics in the generation of estimates for economic analyses.

CHAPTER VIII

ESTIMATION FOR ECONOMIC ANALYSES

The term "estimate," when applied to economic analyses, can have a multiplicity of meanings. On one extreme it can be used to indicate a carefully considered computation of some quantity for which the exact magnitude cannot be determined. On the other extreme, it can be used to denote what are actually just off-hand approximations that are little better than outright guesses.

The basic difficulty of estimating for economic analyses is that most prospective projects for which estimations are to be made are unique; that is, substantially similar projects have not been undertaken in the past under conditions that are substantially the same as expected for the future. Hence, outcome data that can be used in estimating directly and without modification often does not exist. However, it may be possible to gather data on certain past outcomes which are related to the outcomes being estimated, and to adjust and project that data based on expected future conditions. Techniques for collecting and projecting estimation data and also for making probabilistic estimates are rooted in the field of statistics. In this chapter, the use of Bayesian statistics in the making of estimates for economic analyses will be discussed, and recommended procedures to facilitate the estimation of correlation coefficients and parameters of subjective probability distributions will be covered.

Bayesian Statistics

Bayesian statistics is characterized by the adjustment of "prior" subjective probabilities for an unknown parameter or factor to more-reliable "posterior" probabilities based on the results of sample evidence or evidence from further study. Bayes' theorem is frequently employed in this adjustment. Below is shown Bayes' theorem for both the discrete and the continuous cases, and also a simplified procedure for calculating posterior parameters in special cases involving normal distributions.

Bayes' Theorem

Let the outcome of sample evidence, x, be discrete such that $P(x) \neq 0$ and let $Q_1, Q_2, \ldots, Q_j, \ldots, Q_k$ be mutually exclusive, exhaustive outcomes such that $P(Q_j) \neq 0$ for all j. Bayes' theorem for this discrete situation states that:

$$P(Q_j|x) = \frac{P(x|Q_j) \cdot P(Q_j)}{\sum_{j=1}^{k} P(x|Q_j) \cdot P(Q_j)} . \qquad (63)$$

When x has a continuous density function, $f_1(x)$, and Q has a continuous density function, $f_2(Q)$ such that all conditional density functions are continuous, then Bayes' theorem states that

$$f_2(Q\ x) = \frac{f_1(x|Q)f_2(Q)}{\int_{\underline{Q}} f_1(x|Q)f_2(Q)dQ} . \qquad (64)$$

As an example of the use of Bayes' theorem for the discrete case, suppose that it is desired to estimate the expected outcome, $E(Q_j)$, for a project. Prior probability estimates are that there is an even chance that the Q_j, expressed in PV($), will be either $6,000 or $10,000. Thus, the $E(Q_j)$ based on these prior probability estimates is $8,000. It is estimated that a certain amount of added study would result in the following likelihoods: If the Q_j is $6,000, the probability (likelihood) that the added study will have indicated that particular outcome ($P(x = \$6{,}000|Q_j = \$6{,}000)$) is 0.3. If the Q_j is $10,000, the likelihood that the added study will have indicated that particular outcome ($P(x = \$10{,}000|Q_j = \$10{,}000)$) is 0.9. Below is shown a table of computations of posterior probabilities.

Table 15. Computation of Posterior Probabilities--
Example on Use of Bayes' Theorem

True Value of Project Q_j(in PV($))	Prior Probability of Q_j $P(Q_j)$	Likelihood that Added Study Will Indicate Correct Value $P(x = Q_j\|Q_j)$	Joint Probability $P(x,Q_j) = P(Q_j)\cdot(x\|Q_j)$	Posterior Probability $P(Q_j\|x) = \frac{P(x,Q_j)}{P(x)}$
$ 6,000	0.5	0.3	0.15	0.25
10,000	0.5	0.9	0.45	0.75
			$\sum_j = P(x) = 0.60$	

Thus, the calculated posterior probabilities show that there is a 25 per cent chance that the project outcome will be \$6,000 and a 75 per cent chance that the project outcome will be \$10,000. The expected outcome of the project based on these probabilities is \$9,000, which is \$1,000 greater than the expected outcome based on the prior probabilities. If the selection of the project before the added study is mutually exclusive with the selection of the project after the added study, this increase in expected outcome could be considered to be the expected value of the added study and could be compared with the cost of that added study to judge whether that added study is worthwhile.

Determining Posterior Distribution from Prior Distribution When Both Prior and Added Study Distributions are Normal

This section contains a modification of developments by Schlaifer (29-pp.440-448) for the determination of a posterior distribution when prior and sampling distributions are normal and the sampling variance is known. In the usual economic analysis, there is no distribution of outcomes to actually sample from, but rather added study can be made and a subjective probability distribution formulated based on that added study. In this adaptation, Schlaifer's "sampling distribution" is replaced with what will be called an "added study distribution," i.e., a distribution of the estimated outcomes based on the added study. The outcome of any variable or element pertinent to an economic analysis could be considered in this manner.

The same symbols as used by Schlaifer are used in this adaptation and are defined below:

$E_1(\tilde{\mu})$ = mean of posterior distribution,

$E_0(\tilde{\mu})$ = mean of prior distribution,

$\bar{x}$ = mean of distribution based on added study,

$\sigma_1^2(\tilde{\mu})$ = variance of posterior distribution,

$\sigma_0^2(\tilde{\mu})$ = variance of prior distribution, and

$\sigma^2(\tilde{\bar{x}})$ = variance of distribution based on added study.

Schlaifer shows that when both the prior and added study distributions of an outcome are normal with known means and variances, then the posterior distribution of the outcome is normal with the following parameter relations:

$$E_1(\tilde{\mu}) = \frac{E_o(\tilde{\mu})[1/\sigma_o^2(\tilde{\mu})] + \bar{x}[1/\sigma^2(\tilde{\bar{x}})]}{1/\sigma_o^2(\tilde{\mu}) + 1/\sigma^2(\tilde{\bar{x}})} , \tag{65}$$

and

$$1/\sigma_1^2(\tilde{\mu}) = 1/\sigma_o^2(\tilde{\mu}) + 1/\sigma^2(\tilde{\bar{x}}) . \tag{66}$$

As an example of the use of the above adaptation of Schlaifer's developments, suppose that a certain cost element is to be estimated. Prior estimates (i.e., estimates before added study is undertaken) are that the distribution is normal with $E_0(\tilde{\mu})$ = \$7,000 and $\sigma_0^2(\tilde{\mu})$ = 200,000. Results of added study are that the outcome is normally distributed with $\bar{x}$ = \$6,000 and $\sigma^2(\tilde{\bar{x}})$ = 66,667. The posterior distribution resulting from these estimates is normal with the following calculated parameters:

$$E_1(\tilde{\mu}) = \frac{\$7{,}000(1/200{,}000) + \$6{,}000(1/66{,}667)}{1/200{,}000 + 1/66{,}667} = \underline{\underline{\$6{,}250}} \ ;$$

$$\frac{1}{\sigma_1^{\ 2}(\tilde{\mu})} = \frac{1}{200{,}000} + \frac{1}{66{,}667} \ ; \quad \therefore \sigma_1^{\ 2}(\tilde{\mu}) = \underline{\underline{\$50{,}000}} \ .$$

Note from the results of the above example that the posterior mean is closer to the mean based on added study than to the mean of the prior distribution. This is because the variance of the distribution based on the added study is less than the variance of the prior distribution, reflecting the greater confidence in the mean of the distribution based on added study. Note also that the variance of the posterior distribution is less than the variance of either the prior distribution or the distribution based on added study. This is because the combined information of the prior estimates and the estimates based on added study should provide a basis for at least as much confidence as the information of either one of those estimates alone.

To demonstrate the effect of a wide range of conditions on parameters of the posterior distribution, Figures 60 and 61 are shown. Figure 60 shows the behavior of the posterior mean compared to the prior mean for various ratios of the means and the variances, respectively. The ratios of variances that are of greatest interest are those between 0.00 and 1.00, since it is unlikely that the variance of the added study distribution would be greater than the variance of the prior distribution. Figure 62 shows the variance of the posterior distribution as a function of the variances of the prior distribution and the added study distribution.

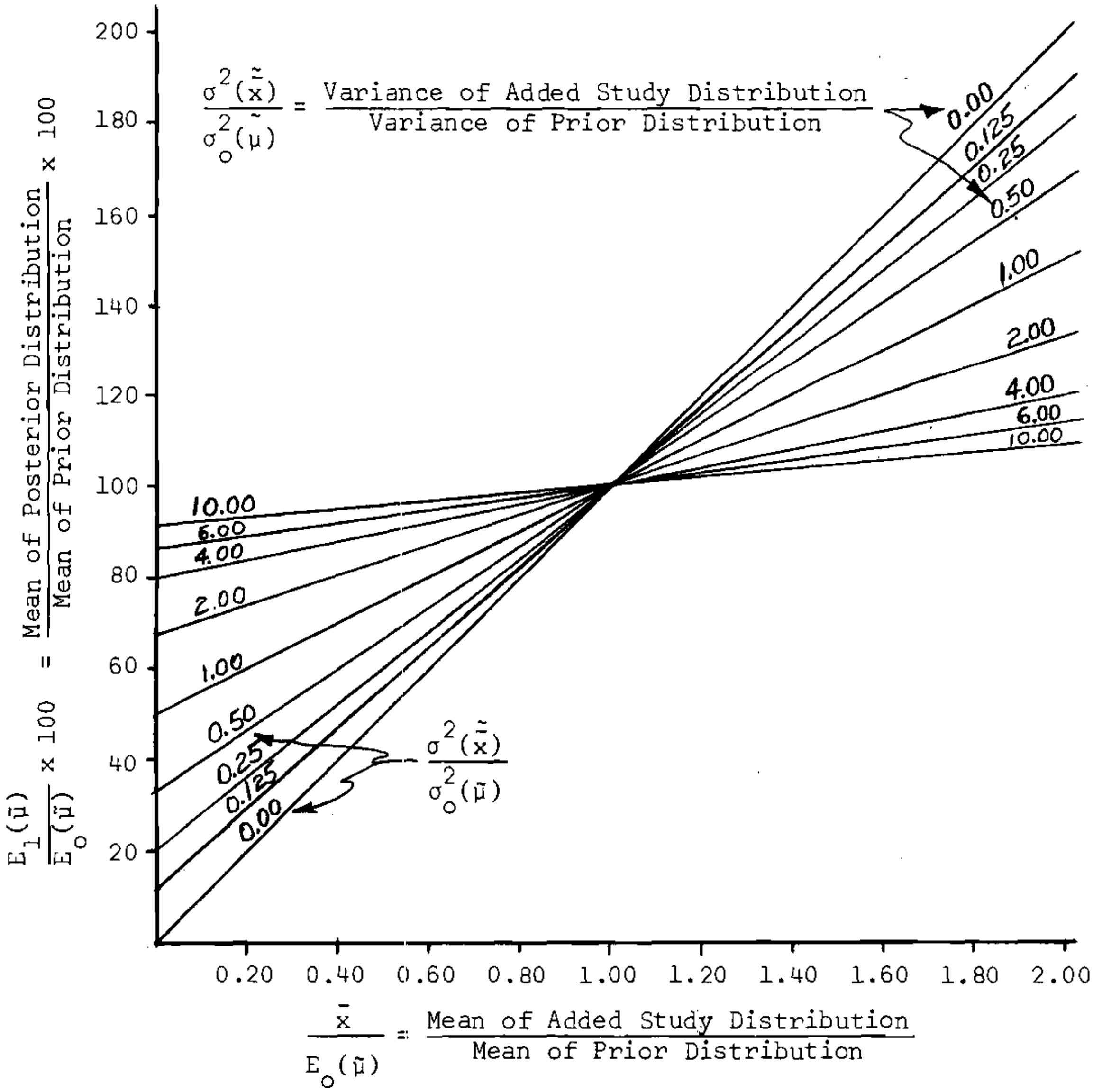

Figure 60. Posterior Mean Expressed as Per Cent of Prior Mean for a Wide Range of $\frac{\bar{x}}{E_o(\tilde{\mu})}$ and $\frac{\sigma^2(\tilde{\bar{x}})}{\sigma_o^2(\tilde{\mu})}$ Conditions

* Graph can be used for any multiple of variances. To adapt, multiply $\sigma_o^2(\tilde{\mu})$ and $\sigma^2(\tilde{\bar{x}})$ [and hence $\sigma_1^2(\tilde{\mu})$] by the same multiple.

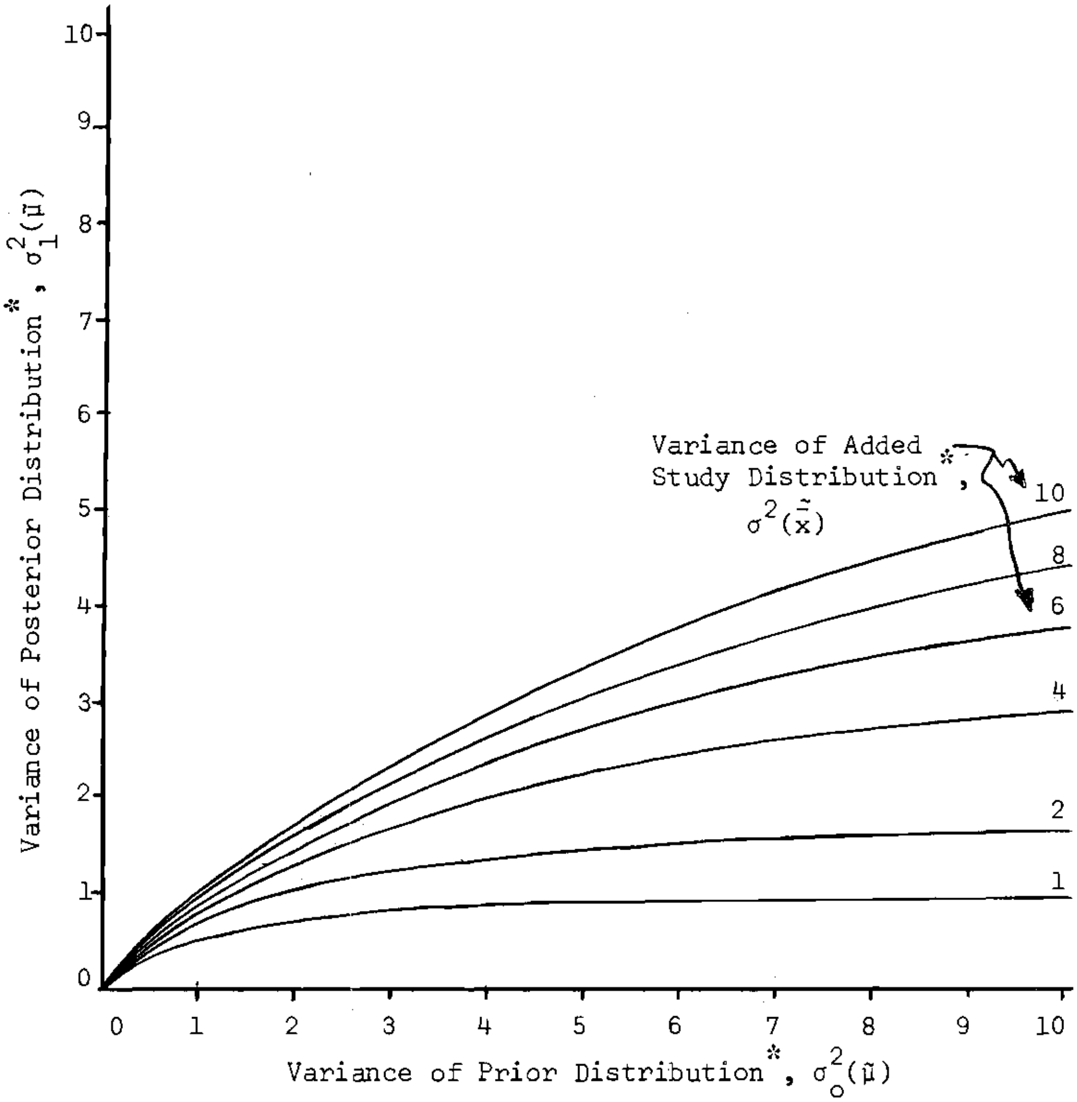

Figure 61. Variance of Posterior Distribution as a Function of Variance of Prior Distribution and Variance of Added Study Distribution

$$\frac{1}{\sigma_1^2(\tilde{\mu})} = \frac{1}{\sigma_o^2(\tilde{\mu})} + \frac{1}{\sigma^2(\tilde{\bar{x}})}$$

The formulas for the above adaptation of Schlaifer's developments strictly apply when both input distributions are normal. However, Schlaifer shows (78-pp.446-448) that if the prior distribution is non-normal, the same formulas can be applied without appreciable loss in accuracy as long as the variance of the prior distribution is large compared to the variance of the distribution based on added study.

Estimation of Correlation Coefficients

There have been several occasions in this thesis in which it was recommended that dependence between two variables be considered by expressing a coefficient of correlation, ρ, for the two variables. This section contains recommendations on procedures for estimating those correlation coefficients.

Suppose that X_1 and X_2 are two random variables with variances, σ_1^2 and σ_2^2, respectively. If the joint probability distribution of X_1 and X_2 is bivariate normal, it can be shown (64-p.203) that:

$$E(X_2|X_1=x) = E(X_2) + \rho \frac{\sigma_2}{\sigma_1}(x-E(X_1)) \qquad (67)$$

The above expression can be manipulated so that X_1 and X_2 are normalized as:

$$\frac{E(X_2|X_1 = x) - E(X_2)}{\sigma_2} = \rho \frac{(x-E(X_1))}{\sigma_1} . \qquad (68)$$

Thus, after normalization, the expected value of X_2, given the value of

X_1, is just ρ times the value of X_1. Expressed in terms of the actual variables, if the value of X_1 lies k standard deviations from its mean, the expected value of X_2 will lie ρk standard deviations from its unconditional mean.

By trying various values of k and estimating the corresponding conditional expected value of X_2, it should be possible to generate a subjective estimate of ρ. Below is an example of how this can be done using just one value of k. Suppose that the parameter estimates for X_1 are $\hat{E}(X_1)$ = \$7,000 and $\hat{\sigma}_1$ = \$1,000; and for X_2 are $\hat{E}(X_2)$ = \$6,500 and $\hat{\sigma}_2$ = \$3,000. For an arbitrary k value of +1.0, the outcome of $X_1 = x$ = \$7,000 + 1.0(\$1,000) = \$8,000. Now, a final estimate required is $E(X_2|X_1$ = \$8,000), which will be supposed to be \$5,600. Using Equation (68):

$$\frac{\$5{,}600 - \$6{,}500}{\$3{,}000} = \rho\frac{(\$8{,}000-\$7{,}000)}{\$1{,}000} .$$

Thus, the estimate of the coefficient of correlation, ρ, can be calculated to be -0.3. For other estimates of ρ, parallel work should be done using other values of k.

If the joint distribution of X_1 and X_2 is not bivariate normal, the expression for $E(X_2|X_1 = x)$ given in Equation (67) does not hold in general. However, this expression does provide the best linear estimate of $E(X_2|X_1 = x)$ according to the principle of least squares.[12] That is, Equation (67), when applied to estimating in non-bivariate normal situa-

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12. See (49-pp.67-69).

tions, results in what is commonly referred to as the mean square regression line of X_2 on X_1. Since it would be very difficult to base a subjective estimate of ρ on a non-linear function, the mean square regression line may well provide a reasonable and practical basis for the estimation.

Estimation with Use of PERT Estimation Procedures

Formal procedures for obtaining an estimated probability distribution for the total time required for a job have been developed for use with PERT network planning techniques. These PERT estimation procedures have achieved considerable publicity for their value in evaluating research, development, and construction program schedules. They are widely accepted in industry even though the theoretical basis behind them is weak.

The PERT procedures are based upon use of the Beta distribution to describe the distribution of times required for a given segment or part of a job. They make use of the central limit theorem and the assumption of independence of the times required for individual job segments so that the estimated total time distribution is normal with easily calculable parameters. This section contains a skeleton description of how these procedures could be adapted for use in estimating for economic analyses.

The use of these procedures for the estimation of a given element or segment of an element (which will be called "variable" in this section) involves first making an "optimistic" estimate, a "pessimistic" estimate, and a "most likely" estimate for the variable. The nature of these estimates should be just what their names imply. It is assumed that these estimates correspond to the lower bound, upper bound, and mode, respectively, of the assumed Beta distribution describing the variable. It is further assumed that the standard deviation of the estimated Beta distribution is equal to 1/6 of the spread between the lower bound and the upper bound.

Under the above assumptions, the approximate mean and variance[13] of the Beta distribution for a given element may be expressed as:

$$E(Y) = \frac{A + 4M + Z}{6}, \tag{69}$$

and

$$V(Y) = \left(\frac{Z-A}{6}\right)^2, \tag{70}$$

where $E(Y)$ = estimated expected outcome,

$V(Y)$ = estimated variance of outcome,

A = estimated pessimistic outcome,

M = estimated most likely outcome, and

Z = estimated optimistic outcome.

If several variables as estimated by the above procedures as assumed to be independent and are added together, the distribution of the total outcome so obtained is approximately normal. The mean of the total outcome distribution can be calculated by adding the means of the individual variables. If the individual variables can be assumed to be in-

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13. See Appendix C for a description of the Beta distribution and a discussion of the nature of the approximations.

dependent, then the variance of the total outcome distribution can be calculated by adding the variances of the distributions of the individual variables.

Subjective Estimation of the Variation of Normally Distributed Elements

Quite often the best subjective estimate of the shape of the distribution of a variable that can be made in practice is that the distribution is normal. This section contains a simple procedure as suggested by Schlaifer (78-pp.438-439) for subjectively determining the variance of a normally distributed variable.

It is easily shown that the middle 50 per cent of a normal distribution is within ± 0.675 standard deviations of the mean of that distribution. Thus, for a normally distributed variable, if one is willing to estimate the smallest range, r, within which that variable is expected to occur with 50 per cent probability, then the standard deviation, σ, for that variable can be calculated by the relation $0.675\sigma = \frac{r}{2}$. In practice, it is probably sufficiently close to approximate the 0.675 with 2/3, as suggested by Schlaifer.

This same idea for estimating the variance for normally distributed variables could be applied using any other number of standard deviations and the associated probability. However, the values suggested above are probably most useful because of the relative ease of visualizing the minimum range which would include 50 per cent probability of occurrence.

This chapter has discussed statistical considerations in the generation of estimates for economic analyses. The use of sampling plans and Bayesian statistics was covered, and procedures for performing subjective estimations were recommended.

CHAPTER IX

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

Expected values of measures of merit based on the explicit consideration of the variation of individual elements in economic analyses can differ notably from parallel measures of merit based on assumed certainty. Relative magnitudes of this difference when variation of one important element, project life, is taken into account is shown in some detail. Ratios are graphed to reflect the relation of the expected value of key factors to those factors under assumed certainty for a wide range of distribution types, distribution variances, interest rates, and expected lives.

Chapter IV shows how analytical consideration of the simultaneous variation of multiple elements in economic analyses can be made through approximations of the expected values and variances of measures of merit. These approximations are based on evaluations of a Taylor series expansion. Procedures are shown for comparing projects when dispersion of multiple elements is considered. These step-by-step procedures are developed for comparing mutually exclusive projects and also non-mutually exclusive projects when covariance between projects is taken into account. A number of figures are shown in Chapter V to aid in calculating certain quantities of interest when comparing pairs of projects.

Consideration of the effect of lack of certainty in the estimation of individual elements is made. Figures are presented to aid the analyst in judging the effect of various degrees of uncertainty for certain conditions by showing the relative effects of various element magnitudes and amounts of variation on the expected value and variance of the measure of merit. Formalized qualitative decision guides are developed to aid in determining the extent to which an economic analysis should be pursued. These guides include consideration of when variation of individual elements should be quantitatively taken into account by risk analyses involving estimated distributions for individual elements and also to what extent lack of certainty concerning those estimated distributions should be examined.

Six extensions of methods for considering random variation of elements in capital investment analyses are described. These methods are rather disjoint and limited in potential application due to their specialized nature. However, they do represent advances which can be significant in direct use or as bases for further development.

Finally, an examination is made of statistical considerations in the generation of estimates for economic analyses. Uses of Bayesian statistics in adjusting probability estimates are shown and procedures for performing subjective estimations are recommended.

Recommendations

The following are recommendations for additional studies which have been generated during this research:

1. The range of conditions considered in the study of the effect

of different life distributions on key factors in economic analyses should be broadened to include other distribution shapes, amounts of variation, and interest rates.

2. Graphical portrayals of the effects of changes in the estimated distributions of individual elements and combinations of elements on the results of project comparisons should be developed for a broad range of conditions. These would be useful to the practitioner in performing sensitivity studies.

3. Further work on effective means of specifying the distributions of outcomes for elements in economic analyses should be done. Particular attention should be paid to means for obtaining information from sources outside the firm; e.g., governmental agencies and companies having experiences with projects similar to those under consideration.

4. A digital computer program for use in general problems which take into account the influence of future alternatives and outcomes on present decisions (e.g., "decision tree" problems) should be developed. This program should include provision for changes in input information such as future alternatives, outcomes, and associated probabilities so that sensitivity studies can be readily made. The program might also include provisions for making Monte Carlo analyses so as to arrive at a measure of the spread or variation as well as the expected value of the outcome for each present alternative.

5. Research and development on the use of analog computing techniques for making economic analyses which include consideration of the variation of individual elements should be performed. No work on analog computers was reported in this research, but it is recognized that they

can be used effectively for sensitivity analyses and for computations involving random inputs (e.g., elements varying according to some distribution).

6. Investigation should be made of the shape of the distribution of measures of merit caused by various combinations of important non-normal distribution shapes for individual elements.

7. Means should be developed for more adequately estimating and taking into account the competitive and/or complementary effects of groups of non-mutually exclusive projects.

8. Explicit investigation should be made of the relative error in approximations of E(PV($)) and V(PV($)) based on the use of a limited number of terms of the Taylor series expansion for a wide range of conditions.

9. Further development of the expectation-variance criterion for use in general economic analyses of projects should be undertaken. Particularly, means of specifying the coefficient of risk aversion when the decision maker's utility of money function is not quadratic should be investigated.

10. Studies of a practical nature should be undertaken to test the useability of the procedures described herein for comparison of both mutually exclusive projects and non-mutually exclusive projects.

APPENDICES

APPENDIX A

DERIVATION OF ESTIMATED EXPECTATION AND VARIANCE OF RESULT VALUE WHEN CONSIDERING EXTERNAL CONDITIONS

Let 0_j ~ j^{th} outcome or state of nature.

x_i ~ i^{th} result value.

$f(0_j, x_i)$ ~ joint density of $(0_j, x_i)$.

$g(x_i | 0_j)$ ~ conditional density of x_i, given 0_j.

$f_j(0_j)$ ~ marginal density of 0_j.

$$E(x_i) = \sum_i x_i \sum_j f(0_j, x_i) \tag{71}$$

$$= \sum_i x_i \sum_j [f_j(0_j) \cdot g(x_i | 0_j)]$$

$$= \sum_i \sum_j x_i \ f_j(0_j) \cdot g(x_i | 0_j)]$$

$$= \sum_j f_j(0_j) [\sum_i x_i \ g(x_i | 0_j)]$$

$$= \sum_j f_j(0_j) \cdot E(x_i | 0_j) \ .$$

$$V(x_i) = \sum_i [x_i - E(x_i)]^2 \cdot f_i(x_i) = \sum_i x_i^2 f_i(x_i) - [E(x_i)]^2 \tag{72}$$

$$= \sum_i x_i^2 \cdot \sum_j [f_j(0_j) \cdot g(x_i | 0_j)] - [E(x_i)]^2$$

$$= \sum_j f_j(0_j) \ [\sum_i x_i^2 \cdot g(x_i | 0_j)] - [E(x_i)]^2$$

$$= \sum_j f_j(0_j) \, [V(x_i|0_j) + \{E(x_i|0_j)\}^2] - [E(x_i)]^2$$

$$= \sum_j f_j(0_j) \cdot V(x_i|0_j) + \sum_j f_j(0_j) \cdot \{E(x_i|0_j)\}^2 - [E(x_i)]^2 \; .$$

APPENDIX B

EFFECT OF VARIOUS SALVAGE VALUE FUNCTIONS AND INTEREST RATES ON CRC_t

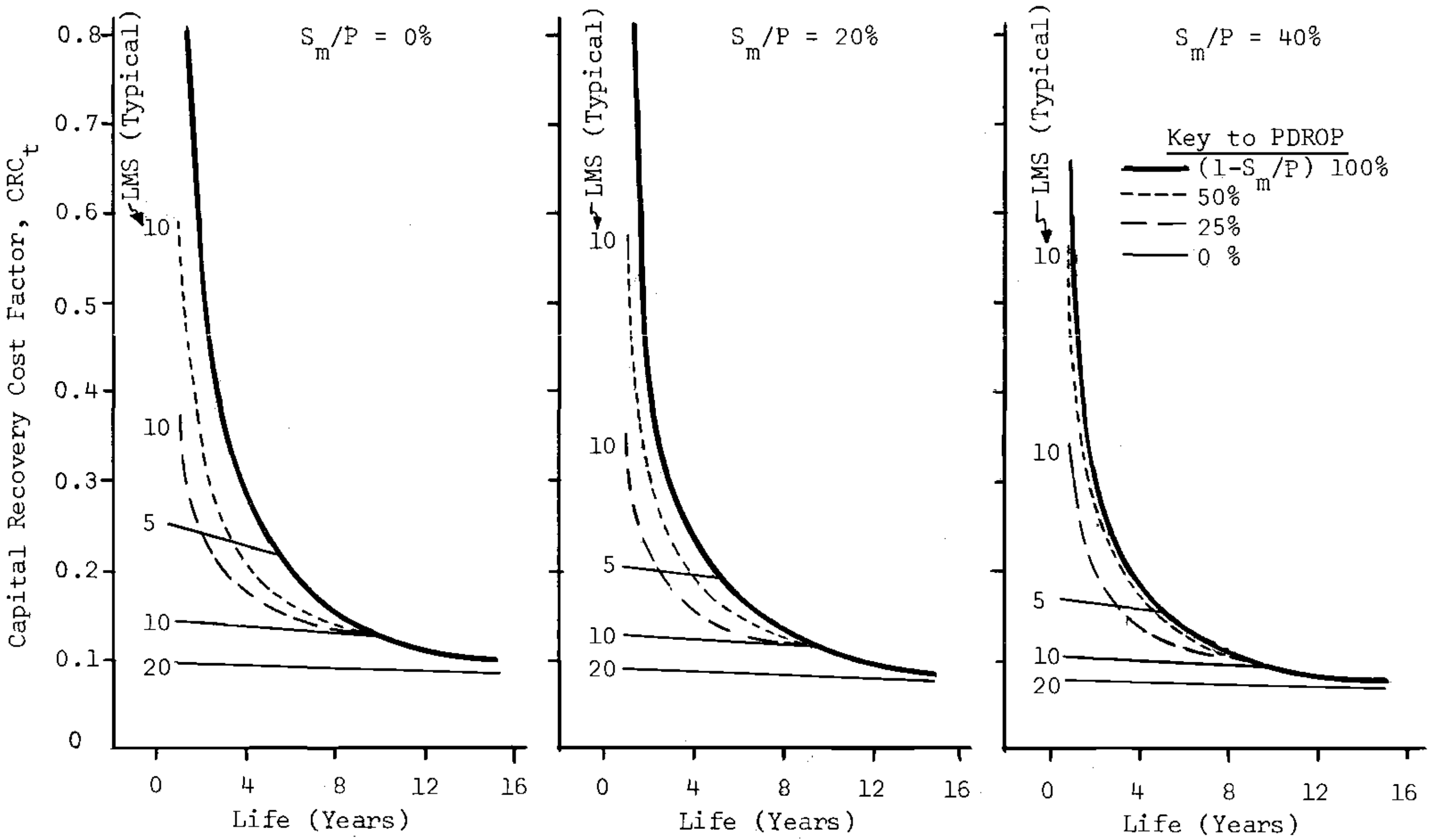

Figure 62. Effect of Straight Line Salvage Value Functions on Capital Recovery Cost Factor for i = 5 Per Cent.

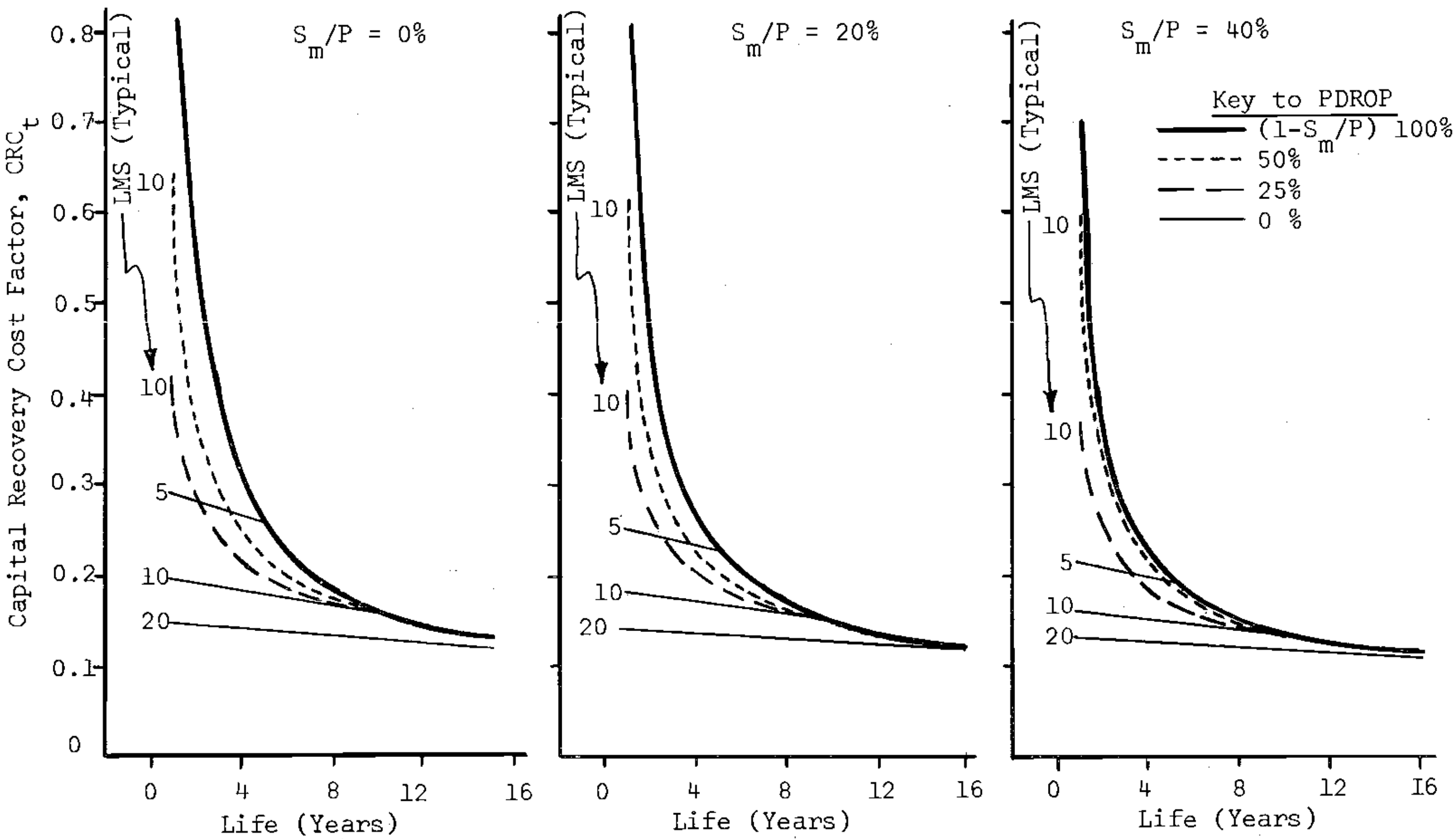

Figure 62. Effect of Straight Line Salvage Value Functions on Capital Recovery Cost Factor for i = 10 Per Cent.

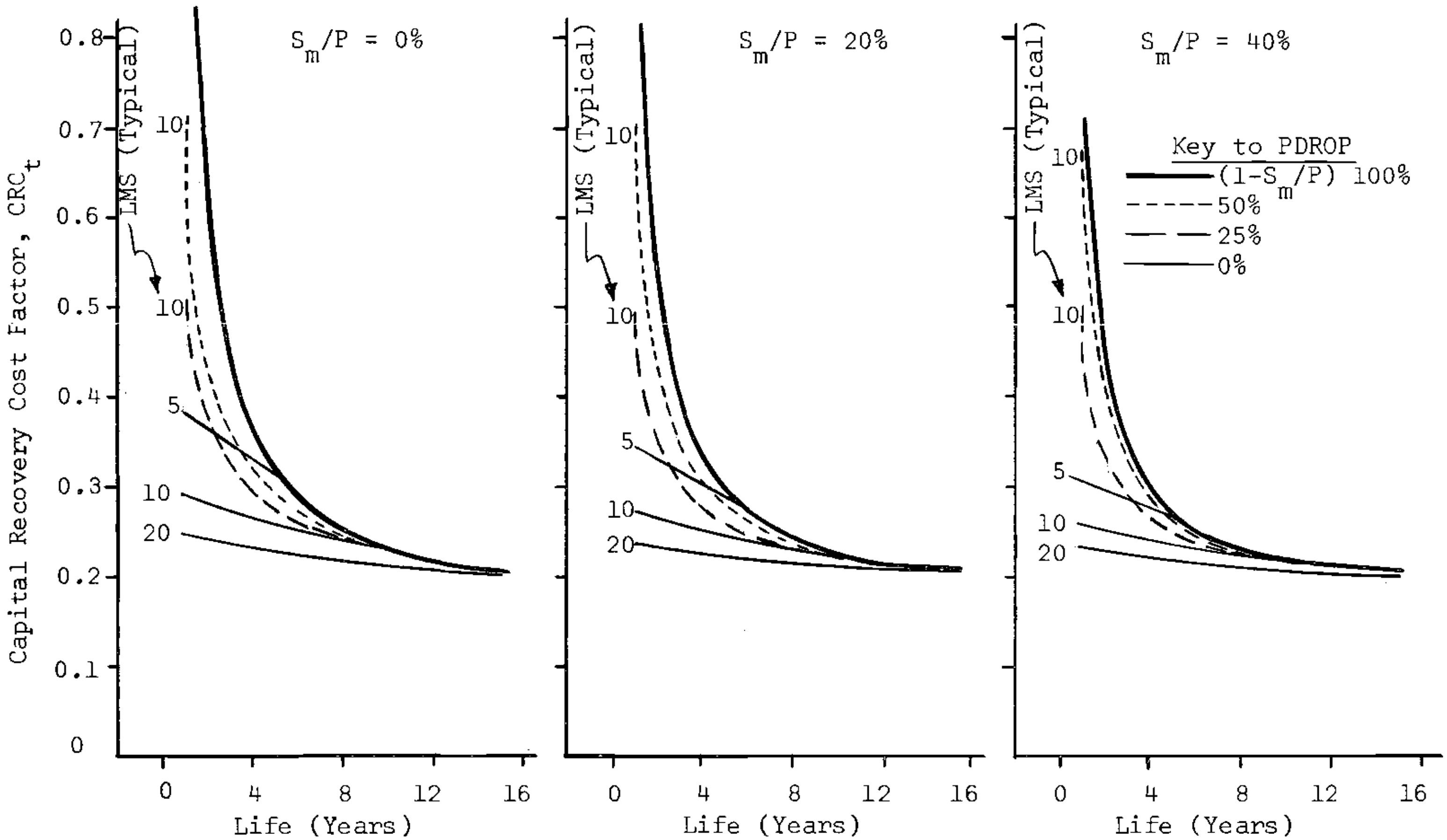

Figure 64. Effect of Straight Line Salvage Value Functions on Capital Recovery Cost Factor for i = 20 Per Cent.

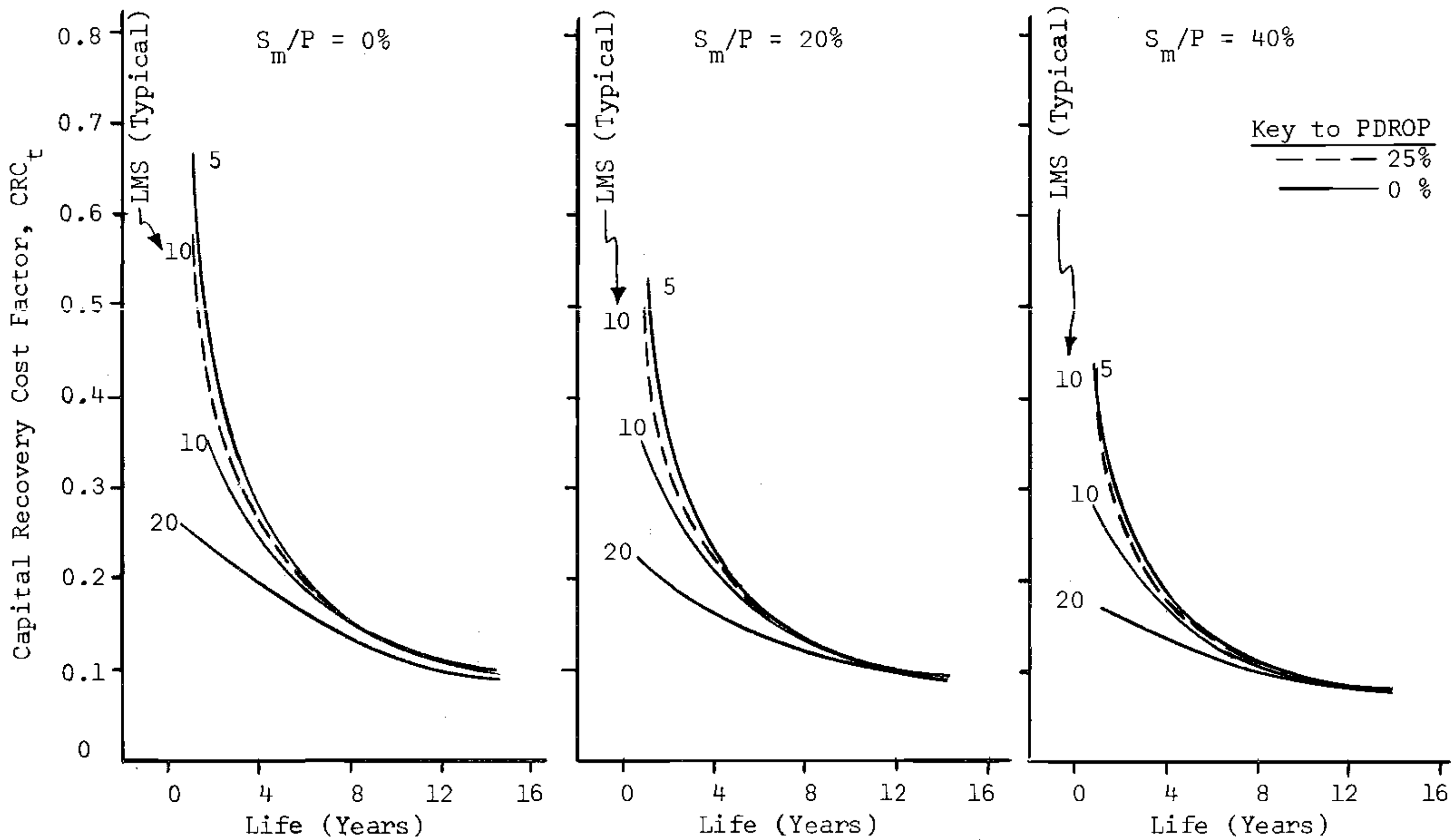

Figure 65. Effect of Exponential Salvage Value Functions on Capital Recovery Cost Factor for i = 5 Per Cent.

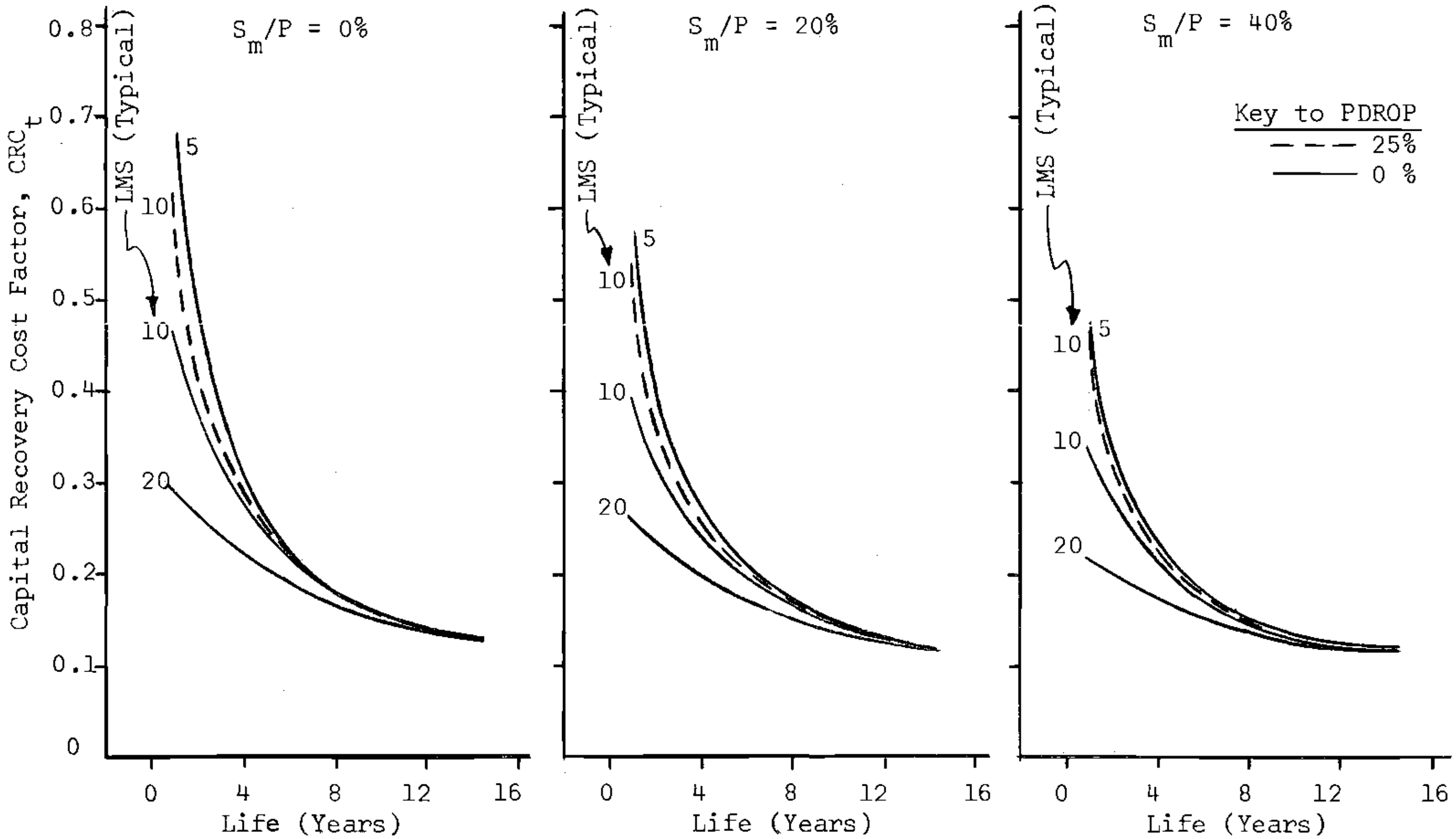

Figure 66. Effect of Exponential Salvage Value Functions on Capital Recovery Cost Factor for i = 10 Per Cent.

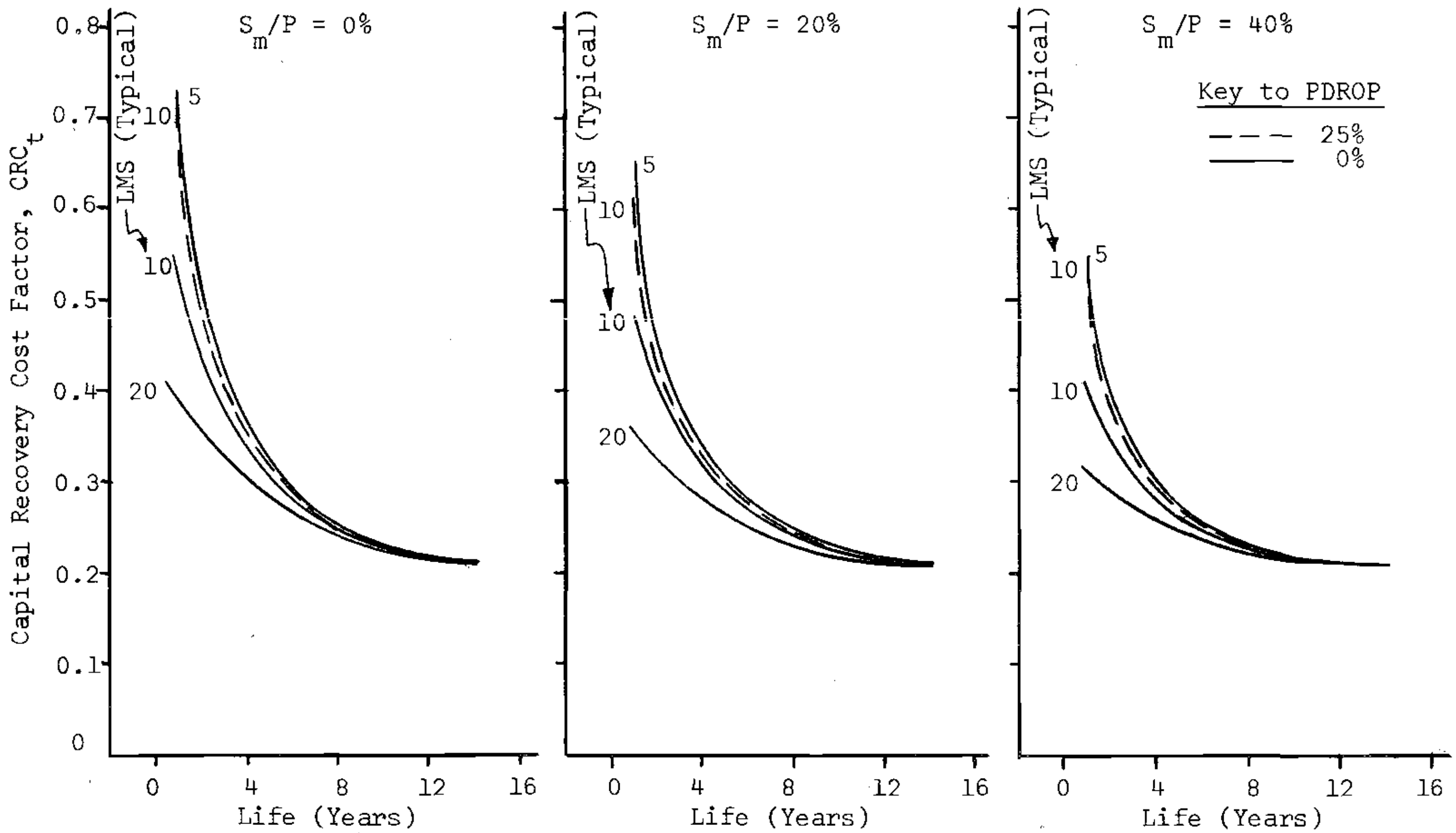

Figure 67. Effect of Exponential Salvage Value Functions on Capital Recovery Cost Factor for i = 20 Per Cent.

APPENDIX C

DERIVATION OF BETA DENSITY FUNCTION AND DISCUSSION OF APPROXIMATIONS MADE IN PERT ESTIMATION PROCEDURES

The Beta density function of the variate x which varies between 0 and 1 is defined as:

$$\beta(x) = \begin{cases} \frac{(\alpha+\beta+1)!}{\alpha!\beta!} x^{\alpha}(1-x)^{\beta}; & 0 \le x \le 1 \\ 0; \text{ otherwise} \end{cases} \tag{73}$$

where α and β are shape parameters.

If a variable, t, varies according to the Beta distribution between the high and low extremes of Z and A, respectively, the distribution of that variable may be obtained by the relation:

$$t = A + (Z-A)\ x \tag{74}$$

Solving the above equation for x and substituting into Equation (73), the result is an equation for the Beta distribution of t:

$$\beta(t) = \begin{cases} \frac{(\alpha+\beta+1)!}{\alpha!\beta!} (\frac{t-A}{Z-A})^{\alpha} (\frac{Z-t}{Z-A})^{\beta}; & A \le t \le Z \\ 0; \text{ otherwise} . \end{cases} \tag{75}$$

Equation (75) can be normalized to form a true density function by dividing by Z-A.

Thus,

$$\beta(t) = \begin{cases} \frac{1}{(Z-A)} \frac{(\alpha+\beta+1)!}{\alpha!\beta!} (\frac{t-A}{Z-A})^{\alpha} (\frac{Z-t}{Z-A})^{\beta}; & A \leq t \leq Z \\ 0; \text{ otherwise} \end{cases} \tag{76}$$

is the Beta density function of the variable, t, which varies between A and Z.

The expected value and the variance of the variable, t, can be shown to be:

$$E(t) = \frac{A+(\alpha+\beta)M+Z}{\alpha+\beta+2} \text{, and} \tag{77}$$

$$V(t) = (Z-A)^2 \quad \{\frac{(\alpha+1)(\beta+1)}{(\alpha+\beta+2)^2(\alpha+\beta+3)}\}, \tag{78}$$

where $M = A+(Z-A)\ [(\alpha-1)/(\alpha+\beta-2)]$ = the modal value of t.

The simplified versions of Equations (77) and (78), as ordinarily used for PERT estimation purposes, are:

$$E(t) \doteq \frac{A+4M+Z}{6} \text{, and} \tag{79}$$

$$V(t) \doteq (\frac{Z-A}{6})^2. \tag{80}$$

These simplified equations are mathematically equivalent to the respective exact expressions shown in Equations (77) and (78) if and only if $\alpha = 2+\sqrt{2}$ and $\beta = 2-\sqrt{2}$, or $\alpha = 2-\sqrt{2}$ and $\beta = 2+\sqrt{2}$, or $\alpha = 3$ and $\beta = 3$. Battersby (5) explains, however, that the difference between the expected values calculated by the exact Equation (77) versus the simplified Equation (79) is relatively small for a wide range of values for α and β. On the other hand, the difference between the variances calculated by the exact Equation (78) versus the simplified Equation (80) can be quite high, and the difference usually is in the direction of underestimation of the exact value.

APPENDIX D

DERIVATION OF TRIANGULAR DENSITY FUNCTION

Let f(t) increase at a constant rate from 0 to A to a maximum at M, and then decrease at a constant rate from the maximum at M to 0 at Z. Thus, $A \leq t \leq Z$, where M is the modal value of t.

In order for a density function to be formed, f(t) evaluated at M must be 2/(Z-A). By taking triangular proportions it can be shown that:

$$f(t) = \begin{cases} 2(t-A)/(Z-A)(M-A); & A \leq t \leq M \\ 2(Z-t)/(Z-A)(Z-M); & M \leq t \leq Z \\ 0; \text{ otherwise.} & \end{cases} \tag{81}$$

As a supplementary note, the expected value of this distribution reduces to the convenient form

$$E(t) = \frac{A+M+Z}{3} . \tag{82}$$

However, the variance has an inconvenient form. If A+M+Z is abbreviated into the symbol S, then

$$V(t) = \frac{1}{9(Z-A)}\left[\frac{4.5M^4-(4S+6A)M^3+(S+6A)SM^2+1.5A^4-(2A^2-SA+2MS)AS}{(M-A)}\right. \tag{83}$$

$$\left. + \frac{1.5Z^4-(2Z^2-SZ+2MS)ZS+4.5M^4-(4S+6Z)M^3+(S+6Z)SM^2}{(Z-M)}\right].$$

APPENDIX E

TAYLOR SERIES EXPANSION FOR APPROXIMATING EXPECTED VALUE AND VARIANCE OF A FUNCTION

The derivation below is shown for the two-variable case. However, the approach can be extended to any number of variables. A function $f(x,y)$ of the random variables x and y can be expanded in a Taylor series around the point vector (x_o,y_o) as:[14]

$$f(x,y) = f(x_o,y_o)+\left[(x-x_o)\frac{\partial f}{\partial x} + (y-y_o)\frac{\partial f}{\partial y}\right]\Big|_0 \tag{84}$$

$$+ \frac{1}{2!}\left[(x-x_o)\frac{\partial f}{\partial x} + (y-y_o)\frac{\partial f}{\partial y}\right]^2\Big|_0 + \ldots + \frac{1}{(n-1)!}\left[(x-x_o)\frac{\partial f}{\partial x} + (y-y_o)\frac{\partial f}{\partial y}\right]^{n-1}\Big|_0 + R$$

where R is the sum of the higher ordered terms of the infinite series, and can be investigated using the relation:[14]

$$R = \frac{1}{n!}\left[(x-x_o)\frac{\partial f}{\partial x} + (y-y_o)\frac{\partial f}{\partial y}\right]^n f[x_o+\theta(x-x_o),y_o+\theta(y-y_o)] \tag{85}$$

where $0 < \theta < 1$.

Let $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ be denoted as $f'x$ and $f'y$, respectively, and let $E(x) = x_o$ and $E(y) = y_o$. If only the first two terms of the expansion are considered, the expected value and the variance of the function can be cal-

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14. For backup, see Kells (53).

culated as shown below

$$E[f(x,y)] = E([f(x_o,y_o)] + E[(x-x_o)f'x + (y-y_o)f'y] \tag{86}$$

$$= f(x_o,y_o)$$

where f'x is evaluated at x_o and f'y is evaluated at y_o. Equation (86) is analogous to the results shown in equation (23).

$$V[f(x,y)] = E[f(x_o,y_o)+(x-x_o)f'x+(y-y_o)f'y-f(x_o,y_o)]^2 \tag{87}$$

$$= V(x)\cdot(f'x)^2 + V(y)\cdot(f'y)^2 + 2Cov(x,y)(f'x)(f'y)$$

where V denotes variance and Cov denotes covariance. If x and y are independent, equation (87) becomes

$$V[f(x,y)] = V(x)\cdot(f'x)^2 + V(y)\cdot(f'y)^2 \tag{88}$$

which is analogous to equation (26).

Results for E[f(x,y)] and V[f(x,y)] are shown above as equalities, but they are actually only approximations since they are based on only the first two terms of the Taylor series expansion. In general, these approximations are good if the coefficients of variation for each of the variables are not too high.

If the coefficients of variation are high, say, greater than 0.2 to 0.3, the number of terms of the expansion which might have to be included in the evaluation for E[f(x,y)] and V[f(x,y)] in order to achieve a given level of accuracy might have to be more than two as shown. The volume of calculations involved in using more than two terms of the expansion for multiple variables can quickly become prohibitive. For example, if the first three terms of the expansion are used in the evaluation of E[f(x)], the results are:

$$E[f(x,y)] = E[f(x_o,y_o)] + 0 + \frac{1}{2!} E[(x-x_o)^2(f'x)^2+(y-y_o)^2(f'y)^2 + (x-x_o)(y-y_o)(f'x)(f'y)] \tag{89}$$

$$= f(x_o,y_o) + \frac{1}{2} [V(x)\cdot(f'x)^2+V(y)\cdot(f'y)^2+Cov(x,y)(f'x)(f'y)].$$

The use of the first three terms of the expansion for the evaluation of V[f(x,y)] can be set up as:

$$V[f(x,y)] = E\{(x-x_o)(f'x)+(y-y_o)(f'y) + \frac{1}{2}[(x-x_o)^2(f'x)^2+(x-x_o)(y-y_o)(f'x)(f'y) + (y-y_o)^2(f'y)^2] - \frac{1}{2}[V(x)\cdot(f'x)^2+V(y)(f'y)^2 + Cov(x,y)(f'x)(f'y)]\}^2 \tag{90}$$

which can be reduced to an expression containing 16 terms. One can readily see how the inclusion of more terms of the Taylor Series expansion can result in numerical chaos, especially in the evaluation of the variance.

Detailed analyses of the accuracy of these approximations for E[f(x,y)] and V[f(x,y)] for particular conditions can be made through computer simulation. The simulations can be used to determine to almost any degree of accuracy what $E(f(x,y))$ and $V(f(x,y))$ should be, and these values can be compared with the respective approximations based on the use of the Taylor series expansion.

APPENDIX F

DERIVATION OF EXPECTED OPPORTUNITY LOSS FOR USE IN EVALUATING PROJECTS WHERE DISTRIBUTION OF OUTCOMES IS NORMAL[15]

The expected opportunity loss (called EOL and also variously known as expected cost of uncertainty and expected value of perfect information) for the outcome of a given project or for the difference between two projects may be stated in general as:

$$\mathrm{EOL} = \int_{-\infty}^{Q_b} (Q_b - Q)\cdot f(Q)\, dQ \qquad (91)$$

where Q is outcome of the project or the difference between two projects (expressed in a measure like PV($) or AW($)); Q_b is the breakeven outcome (usually zero); and the expected value of $Q \geq Q_b$. Thus,

$$\mathrm{EOL} = Q_b \int_{-\infty}^{Q_b} f(Q)\, dQ - \int_{-\infty}^{Q_b} Q\cdot f(Q)\, dQ \qquad (92)$$

where f(Q) is the density function of Q. If Q is normally distributed, Schlaifer (78-p.453) shows that:

$$\int_{-\infty}^{Q_b} Q\cdot f(Q)\, dQ = E(Q) \int_{-\infty}^{Q_b} f(Q)\, dQ - \sigma(Q)\cdot f(Q_b) \qquad (93)$$

15. This is based on Robert Schlaifer's work (78-pp.452-453).

where $E(Q)$ and $\sigma(Q)$ denote the expected value and standard deviation, respectively, of Q. Substituting Equation (93) into Equation (92).

$$EOL = Q_b \int_{-\infty}^{Q_b} f(Q)\ dQ - E(Q) \int_{-\infty}^{Q_b} f(Q)\cdot dQ + \sigma(Q)\cdot f(Q_b)\ . \tag{94}$$

By manipulation and regrouping of terms,

$$EOL = \sigma(Q)[f(Q_b) - \frac{E(Q)-Q_b}{\sigma(Q)} \int_{-\infty}^{Q_b} f(Q)\ dQ]\ . \tag{95}$$

The quantity in the brackets is tabled by Schlaifer (78-pp.706-707) and called the "unit normal loss integral at u," where $u = (E(Q)-Q_b)/\sigma(Q)$. Thus,

$$EOL = \sigma(Q) \cdot \text{Unit Normal Loss Integral at } u. \tag{96}$$

APPENDIX G

SAMPLE COMPUTER PROGRAMS

(Written in Algol for use on the Burroughs 220 machine)

Table 16. Computer Program for Sensitivity Analysis

```
2 COMMENT INVESTMENT RISK ANALYSIS, JR CANADA, IE  $
2 INTEGER  P,SP, S,SS, D, SD  $
2 BOOLEAN Z  $
2 INPUT INA(I,P,SP,S, SS, D, SD, T, ST, ROTD, ROTS)  $
2 OUTPUT ANSA(I,P,SP,S,SS,D,SD,T,ST,ROTD,ROTS,EAW,SAW,STOLOSS, K, AWK) $
2 FORMAT FMT(X5.2, B1, 6I8, B1, 4X6.2, 2X9.1,  2X5.2, X10.1, W2) $
2 LP1.. READ($Z$INA)  $
2 NEX = EXP(-I.T)  $
2 CRF  = I/(1-NEX)   $
2 PARP = CRF         $
2 PARS =+CRF -I    $
2 PART= -(P+S)((CRF)*2)(NEX)  $
2 PARD = 1            $
2 SECPART = ((I*3).(NEX).(1+NEX))/((1-NEX)*3)  $
2 ECRF = CRF +((SECPART).(ST)*2)/2  $
2 EAW =  (P+S)(ECRF) -S.I +D  $
2 VAW  = (PARP.SP)*2 +(PARS.SS)*2 +(PARD.SD)*2 +(PART.ST)*2
2          +2.ROTD.ST.SD +2.ROTS.ST.SS  $
2 SAW  = SQRT(VAW) $
2 STOLOSS = -EAW/SAW  $
2 K = -2.0  $
2 LP2..
2      KP = P+K.SP          $
2     KS = S+K.SS  $
2     KD = D+K.SD  $
2     KT = T+K.ST  $
2 AWK = (KP+KS).I/(1-EXP(-I.KT)) -KS.I +KD   $
2 WRITE($$ANSA,FMT)  $
2 IF K EQL 2.0  $  GO TO LP1  $
2 K = K +0.5  $
2 GO TO LP2  $
2 FINISH  $
```

Table 17. Computer Program for Determining Effect of Straight Line Salvage Functions on CRC_t

```
2 COMMENT EFFECT OF ST. LINE SALVAGE FCNS. ON CR COST,JR CANADA,IE  $
2 REAL I,PDROP, LMS, DECR, PS, CRF, CRC, T  $
2 BOOLEAN Z   $
2 INPUT INA(I, PDROP, LMS, PS)  $
2 OUTPUT ANSA(I, PDROP, LMS, DECR, PS(  CRF, CRC, T)  $
2 FORMAT FMT(3X9.3, 4F12.5, X7.2, W2)
2 LP1..READ($7$INA)  $
2 T = 2.0   $
2 LP4..DECR = (1-PS-PDROP)/LMS    $
2 CRF = I/(1-EXP(-I.T))  $
2 IF (T LSS LMS)$ GO TO LP2  $
2 GO TO LP3  $
2  LP2..SALV = 1 -PDROP -DECR.T   $
2        CRC =(1-SALV)(CRF) +(SALV)(I)  $
2 GO TO LP5  $
2 LP3.. CRC =(1-PS)(CRF)+PS(I)  $
2 LP5..WRITE($$ANSA,FMT)  $
2 IF T EQL 4   $
2 BEGIN
2 GO TO LP1
2 END  $
2 T=T+1.0     $
2 GO TO LP4  $
2 FINISH  $
```

Table 18. Computer Program for Calculating R_{cr} and R_{crm} by Gaussian Mechanical Quadrature Method

```
2 COMMENT EXP. CRF WITH TRIANG. DISTRIB., JR CANADA, IF 12-2-64   $
2  BOOLEAN Z      $
2 INTEGER M    $
2 FUNCTION FCN11(T) = (T-TMIN)/((1-EXP(-I.T))(M-TMIN))    $
2 FUNCTION FCN22(T) = (TMAX-T)/((1-EXP(-I.T))(TMAX-M))    $
2ARRAY   AR3(16)  $
2ARRAY  AR1(15,16) =(
2   1.0,
2   1.0,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2   0.55555556,   0.88888889, 0.55555556,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2   0.34785485, 0.65214515, 0.65214515, 0.34785485,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2   0.23692689, 0.47862867,   0.56888889, 0.47862867, 0.23692689,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2   0.17132449, 0.36076157, 0.46791393, 0.46791391, 0.36076157,
2   0.17132449,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2   0.12948497, 0.27970539, 0.38183005, 0.41795918,
2   0.38183005, 0.27970539 , 0.12948497,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2   0.10122854, 0.22238103, 0.31370665, 0.36268378,
2   0.36268378, 0.31370665    , 0.22238103, 0.10122854,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2   0.08127439, 0.18064816, 0.26061070, 0.31234708, 0.33023936,
2   0.31234708, 0.26061070, 0.18064816, 0.08127439,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2   0.06667134, 0.14945135, 0.21908636, 0.26926672, 0.29552422,
2   0.29552422, 0.26926672, 0.21908636, 0.14945135, 0.06667134,
2   0.0,0.0,0.0,0.0,0.0,0.0,
2   0.05566857, 0.12558037, 0.18629021, 0.23319376, 0.26280454,
2   0.27292509, 0.26280454, 0.23319376, 0.18629021,
2   0.12558037, 0.05566857,
2   0.0,0.0,0.0,0.0,0.0,
2    0.04717534,    0.10693933, 0.16007833,  0.20316743,
2   0.23349254,   0.24914705, 0.24914705,  0.23349254,   0.20316743,
2    0.16007833,  0.10693933,   0.04717534,
2   0.0,0.0,0.0,0.0,
2   0.04048400, 0.09212150, 0.13887351, 0.17814598,
2   0.20781605, 0.22628318, 0.23255155, 0.22628318, 0.20781605,
2   0.17814598, 0.13887351, 0.09212150, 0.04048400,
2   0.0,0.0,0.0,
2   0.03511946, 0.08015809, 0.12151857, 0.15720317, 0.18553840,
2   0.20519846, 0.21526385, 0.21526385, 0.20519846, 0.18553840,
2   0.15720317, 0.12151857, 0.08015809, 0.03511946,
2   0.0,0.0,
2   0.03075324, 0.07036605, 0.10715922, 0.13957068, 0.16626921,
2   0.18616100,  0.19843149, 0.20257824, 0.19843149, 0.18616100,
2   0.16626921, 0.13957068, 0.10715922, 0.07036605, 0.03075324,
2   0.0,
2   0.02715246, 0.06225352, 0.09515851, 0.12462897,
2   0.14959599, 0.16915652, 0.18260342, 0.18945061,
2  0.18945061, 0.18260342, 0.16915652, 0.14959599,
2   0.12462897, 0.09515851, 0.06225352, 0.02715246  )      $
2ARRAY  AR2(15,16) = (
2  -0.57735027, 0.57735027,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2   -0.77459667, 0.0, 0.77459667,
2  0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2  -0.86113631, -0.33998104, 0.33998104, 0.86113631,
2  0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
```

Table 18. Computer Program for Calculating R_{cr} and R_{crm} by Gaussian Mechanical Quadrature Method (Continued)

```
2  -0.90617985, -0.53846931, 0.0, 0.53846931, 0.90617985,
2  0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2  -0.93246951,-0.66120939, -0.23861919,
2   0.23861919, 0.66120939, 0.93246951,
2  0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2  -0.94910791, -0.74153119, -0.40584515, 0.0    ,
2  0.40584515, 0.74153119, 0.94910791 ,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2   -0.96028986, -0.79666648, -0.52553241, -0.18343464,
2   0.18343464, 0.52553241, 0.79666648, 0.96028986,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2  -0.96816024, -0.83603111, -0.61337143, -0.32425342, 0.0,
2        0.32425342, 0.61337143, 0.83603111, 0.96816024,
2   0.0,0.0,0.0,0.0,0.0,0.0,0.0,
2   -0.97390653, -0.86506337, -0.67940957, -0.43339539, -0.14887434,
2   0.14887434, 0.43339539, 0.67940957, 0.86506337, 0.97390653,
2  0.0,0.0,0.0,0.0,0.0,0.0,
2   -0.97822866, -0.88706260, -0.73015201, -0.51909613, -0.26954316,
2   0.0, 0.26954316, 0.51909613, 0.73015201, 0.88706260, 0.97822866,
2  0.0,0.0,0.0,0.0,0.0,
2  -0.98156063, -0.90411726, -0.76990267, -0.58731795,
2  -0.36783150, -0.12523341, 0.12523341, 0.36783150,
2  0.58731795,  0.76990267,  0.90411726, 0.98156063, 0.0,0.0,0.0,0.0,
2  -0.98418305, -0.91759840, -0.80157809, -0.64234934, -0.44849275,
2  -0.23045832, 0.0, 0.23045832, 0.44849275, 0.64234934,
2  0.80157809, 0.91759840, 0.98418305,
2   0.0,0.0,0.0,
2  -0.98628381, -0.92843488, -0.82720132, -0.68729290, -0.51524864,
2  -0.31911237, -0.10805495, 0.10805495, 0.31911237 ,
2  0.51524864,0.68729290, 0.82720132, 0.92843488, 0.98628381,
2   0.0,0.0,
2  -0.98799252, -0.93727339,-0.84820658, -0.72441773, -0.57097217,
2  -0.39415135,  -0.20119409,  0.0, 0.20119409, 0.39415135,
2  0.57097217, 0.72441773, 0.84820658, 0.93727339, 0.98799252,
2  0.0,
2  -0.98940093, -0.94457502,  -0.86563120,  -0.75540441, -0.61787624,
2  -0.45801678,  -0.28160355,  -0.09501251,  0.09501251,
2  0.28160355, 0.45801678,  0.61787624, 0.75540441,  0.86563120,
2  0.94457502,  0.98940093    )    $
2 START.. READ($Z$INA)      $
2 INPUT INA(I, G, HH)  $
2 M = 1    $
2 CYCLE.. TMIN = M-(G.M)   $
2 TMAX = M +(HH.M)   $
2 EXPL =  (TMIN+TMAX+M)/3.0          $
2          H=(M-TMIN)/25    $ TA=TMIN-H   $ INTER11=0.0   $
2          FOR K=(1,1,25)   $
2 BEGIN    TA=TA +H   $
2          TB=TA +H   $
2           AA=(TB-TA)/2.0   $  BB=(TB+TA)/2.0   $
2          N=1   $
2 LA..     N=N+1 $  SUM=0.0   $
2          IF N GTR 16   $  GO TO LC   $
2          FOR J=(1,1,N)   $
2 BEGIN    TRM = (AA)(AR2(N-1,J)) +BB    $
2          SUM = SUM + AR1(N-1,J).FCN11(TRM)     END     $
2          AR3(N) = SUM $
2          IF N LSS 3 $  GO LA   $
2          TSTA = ABS((AR3(N)-AR3(N-1))/(AR3(N)))   $
2          IF TSTA GTR 0.00001 $  GO LA   $
2          INTER11 = INTER11 + (AR3(N))(H/2.0)   $
2          END  $
```

Table 18. Computer Program for Calculating R_{cr} and R_{crm} by Gaussian Mechanical Quadrature Method (Continued)

```
2          H=(TMAX-M)/25    $ TA=M-H      $ INTER22=0.0  $
2          FOR K=(1,1,25)   $
2 BEGIN    TA=TA +H  $
2          TB=TB +H  $
2           AA=(TB-TA)/2.0  $  BB=(TB+TA)/2.0  $
2          N=1  $
2 KA..     N=N+1 $  SUM=0.0  $
2          IF N GTR 16  $  GO TO LC  $
2          FOR J=(1,1,N)  $
2 BEGIN    TRM = (AA)(AR2(N-1,J)) +BB    $
2          SUM = SUM + AR1(N-1,J).FCN22(TRM)     END    $
2          AR3(N) = SUM $
2          IF N LSS 3 $  GO KA  $
2          TSTA = ABS((AR3(N)-AR3(N-1))/(AR3(N)))  $
2          IF TSTA GTR 0.00001 $  GO KA  $
2          INTER22 = INTER22 + (AR3(N))(H/2.0)  $
2          GO TO LB  $
2 LC..     WRITE($$TITL1)  $
2 LB..     END  $
2 EXPCRF =(2.I/(TMAX-TMIN))(INTER11 + INTER22)  $
2 CERCRF = I/(1-EXP(-I.EXPL))  $
2 MODCRF = I/(1-EXP(-I.M))  $
2 RATEXPCRF = EXPCRF/CERCRF  $
2 RATMODCRF = EXPCRF/MODCRF  $
2 WRITE($$ANSA,FMT)  $
2 OUTPUT ANSA(I,G,HH,M,EXPL,EXPCRF,CERCRF,MODCRF,RATEXPCRF,RATMODCRF) $
2 FORMAT FMT(3X12.2, B2, I3,B2,6F12.5, W2)      $
2 FORMAT TITL1(B5,*DID NOT CONVERGE*,W2)     $
2 IF M EQL 32 $
2 GO TO START          $
2 M=2.M $
2 GO TO CYCLE $
2 FINISH  $
```

Table 19. Computer Program for Determining Relative Effect of Variation on E(AW($)) and $\sqrt{V(AW(\$))}$

```
2 COMMENT EFFECTS OF VARIATION IN ELEMENTS OF INV  JR CANADA, IE  $
2 INTEGER P, S   $
2 BOOLEAN Z  $
2 INPUT INA(I, P, S, T)   $
2 OUTPUT ANSA(I, P, S, T, M, D, EAW, SAW, CAW, REAWCAW, RSAWCAW)  $
2 FORMAT FMT(X5.2, B1, 2I10,B1, 2X9.2, B2, 4X9.1, 2X7.3,W2)  $
2 LP1..READ($Z$INA) $
2 CCRC =(P+S)(I/(1-EXP(-I.T))) -S.I    $
2 M =0  $
2 LP2.. D = M.CCRC  $
2 CAW = CCRC +D  $
2 CV = 0.0 $
2 LP3..SP = CV.P  $
2       SS = CV.S  $
2       SD = CV.D  $
2       ST = CV.T  $
2 NEX = EXP(-I.T) $
2 SECPART = ((I*3).(NEX).(1+NEX))/((1-NEX)*3)  $
2 ECRF = I/(1-NEX) +((SECPART).(ST)*2)/2  $
2 EAW = (P+S)(ECRF) -S.I +D  $
2 CRF =I/(1-NEX) $
2 PARP = CRF $
2 PARS = CRF -I   $
2 PART = -(P+S)((CRF)*2)(NEX) $
2 PARD = 1   $
2 VAW = (PARP.SP)*2 +(PARS.SS)*2 +(PARD.SD)*2 +(PART.ST)*2   $
2 SAW = SQRT(VAW) $
2 REAWCAW = EAW/CAW  $
2 RSAWCAW = SAW/CAW  $
2 WRITE($$ANSA,FMT)   $
2 IF CV EQL 0.3  $  GO TO LP4  $
2 CV = CV +0.1  $
2 GO TO LP3  $
2 LP4..IF M EQL 2.0 $  GO TO LP1  $
2 M = M +0.5  $
2 GO TO LP2  $
2 FINISH  $
```

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>
> "Capital Budgeting: Its Nature, Present, and Needs for the Future," *Journal of Industrial Engineering*, March-April, 1964.
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